Math 20300
Calculus III

Lesson 14
Partial Derivatives

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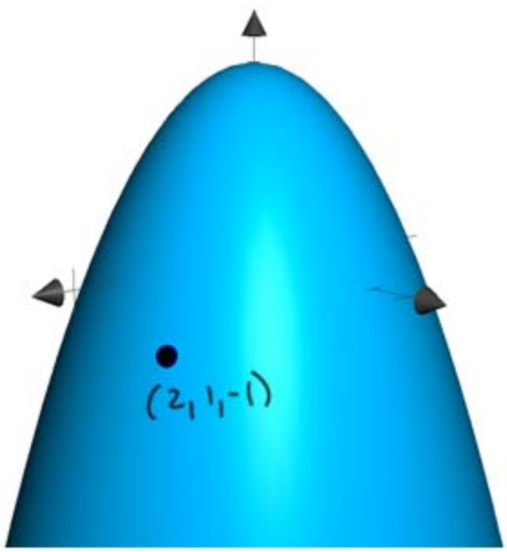
Partial Derivatives

For $y = f(x)$, $f'(x_0)$ gives us the slope of the graph of f at $x = x_0$.

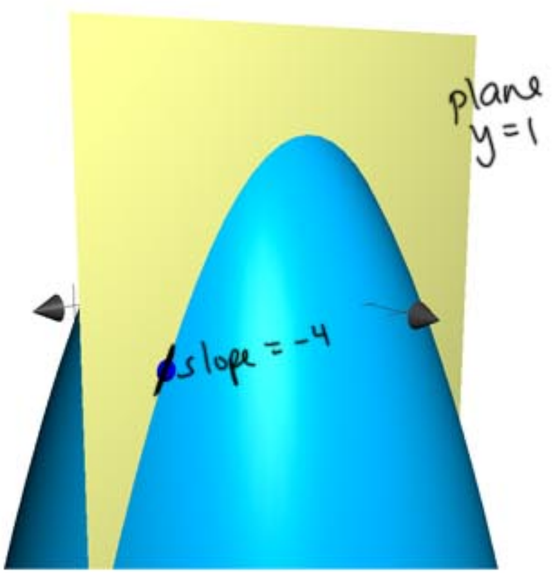
But for $z = f(x,y)$ what would "the slope" mean?

On a surface, different slopes in different directions. We'll start by looking at the slope in the x-direction and the slope in the y-direction. (Slopes in other directions can be found using these.)

Ex. Consider $f(x,y) = -x^2 - y^2 + 4$ and point $(2,1,-1)$ on the paraboloid



To consider the slope in the x-direction, we treat y as a constant (here $y = 1$ since we're at $(2,1,-1)$):



take the slope of the curve of intersection of the surface and the plane to do this we take a <u>partial derivative</u>

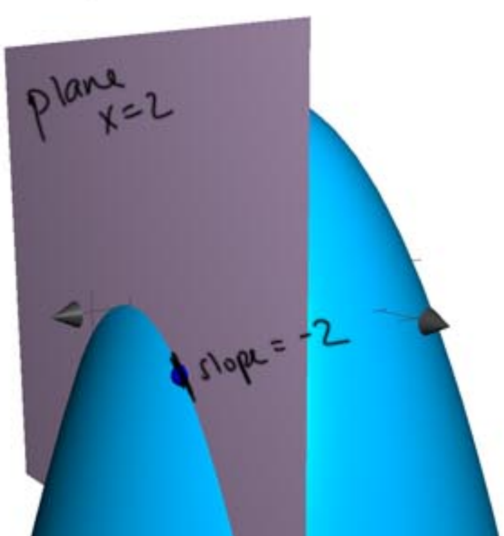
<u>in the x-direction</u>, ie we treat y as a constant and differentiate with respect to x :

$$f(x,y) = -x^2 \overbrace{-y^2}^{\text{treat as constant}} \overbrace{+4}^{\text{constant}}$$

$$f_x(x,y) = -2x$$

$$\text{so } f_x(2,1) = -2(2) = -4$$ ← slope in the x-direction at $(2,1,-1)$

Similarly for the slope in the y direction, we treat x as a constant, and take the



slope of the curve of intersection of the plane (here x=2) and the surface to do this we take a partial derivative in the y-direction, ie we treat x as a constant and take the derivative with respect to y:

$$f(x,y) = \overbrace{-x^2}^{\text{treat as constant}} - y^2 \overbrace{+4}^{\text{constant}}$$

$$f_y(x,y) = -2y$$

$$\text{so } f_y(2,1) = -2(1) = -2$$ ← slope in the y-direction at $(2,1,-1)$

Definitions:

y is fixed

$$f_x(x,y) = \lim_{h \to 0} \frac{f(\underline{x+h}, y) - f(\underline{x}, y)}{h}$$

x is fixed

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, \underline{y+h}) - f(x, \underline{y})}{h}$$

Notation: for $z = f(x,y)$

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{\partial z}{\partial x} = D_x f$$

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = \frac{\partial z}{\partial y} = D_y f$$

Above we computed

$$f_x(2,1) = \frac{\partial f}{\partial x}(2,1) = \left.\frac{\partial f}{\partial x}\right|_{(2,1)} = \left.\frac{\partial z}{\partial x}\right|_{(2,1)}$$

Ex. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = x^2y^3 - 2e^{xy}$

$\frac{d}{dx}(x^2y^3) = y^3 \cdot 2x$
compare to
$\frac{d}{dx}(5x^2) = 5 \cdot 2x$

treat y as a constant $\quad \frac{\partial f}{\partial x} = 2xy^3 - 2ye^{xy}$

$\frac{\partial}{\partial x}(2e^{xy}) = 2 \cdot ye^{xy}$
compare to
$\frac{d}{dx}(2e^{3x}) = 2 \cdot 3e^{3x}$

treat x as a constant $\quad \frac{\partial f}{\partial y} = 3x^2y^2 - 2xe^{xy}$

Ex. Find the slope in the x-direction

along the surface $z = \cos(x^2+y^2)$

at $\underset{(x,y)}{(0, \sqrt{\pi})}$. (could also say at $\underset{(x,y,z)}{(0, \sqrt{\pi}, -1)}$)

find $\left.\dfrac{\partial z}{\partial x}\right|_{(0,\sqrt{\pi})}$

chain rule

$$\frac{\partial z}{\partial x} = -\sin(x^2+y^2)\cdot\frac{\partial}{\partial x}(x^2+y^2)$$

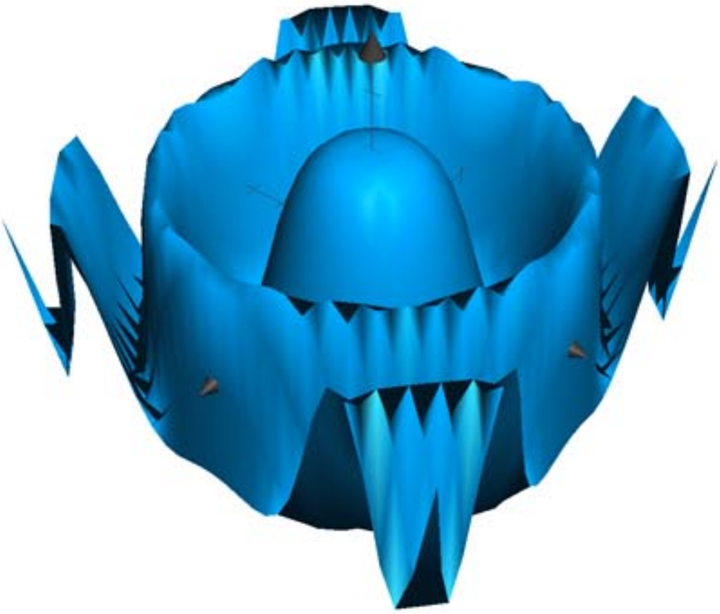
treat y as a constant

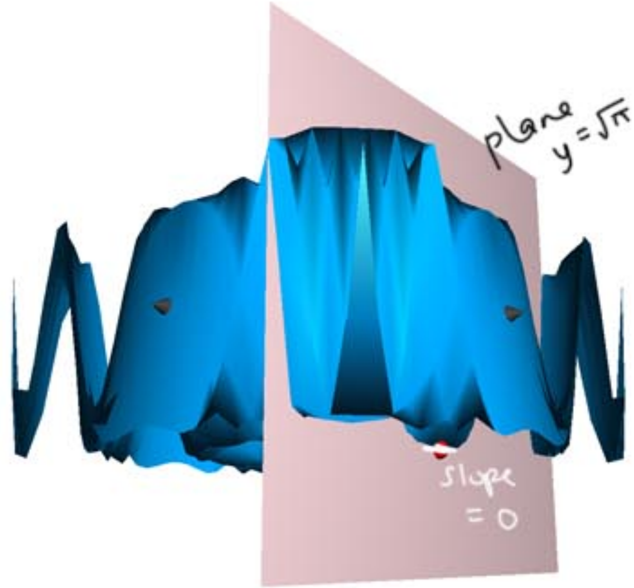
$$= -\sin(x^2+y^2)\cdot 2x$$

$$= -2x\sin(x^2+y^2)$$

$$\left.\frac{\partial z}{\partial x}\right|_{(0,\sqrt{\pi})} = -2(0)\sin(0^2+\pi) = 0.$$

$z = \cos(x^2+y^2)$





Ex. Find $\left.\dfrac{\partial z}{\partial y}\right|_{(2,-3)}$ for $z = y\ln(x^2+y^2)$

treat x as a constant

$$\frac{\partial z}{\partial y} = \underbrace{\frac{\partial}{\partial y}(y)\cdot\ln(x^2+y^2) + y\cdot\frac{\partial}{\partial y}\left(\ln(x^2+y^2)\right)}_{\text{product rule}}$$

$$= 1\cdot\ln(x^2+y^2) + y\cdot\underbrace{\frac{1}{x^2+y^2}\frac{\partial}{\partial y}(x^2+y^2)}_{\text{chain rule}}$$

$$= \ln(x^2+y^2) + \frac{y}{x^2+y^2}\cdot 2y$$

$$= \ln(x^2+y^2) + \frac{2y^2}{x^2+y^2}.$$

then $\left.\frac{\partial z}{\partial y}\right|_{(2,-3)} = \ln(4+9) + \frac{2(9)}{4+9} =$

$$= \ln(13) + \frac{18}{13}.$$

For functions of several variables,

$w = f(x,y,z)$ we'll have $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$

or $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

or f_x, f_y, f_z

for $\frac{\partial w}{\partial x}$, treat y and z as constants.

for $\frac{\partial w}{\partial y}$, treat x & z as constants

for $\frac{\partial w}{\partial z}$, treat x & y as constants

Ex. $f(x,y,z) = 2xy^2 - 3yz^5 + 17x^2yz$

$\frac{\partial f}{\partial x} = 2y^2 + 34xyz$ $\qquad$ $\frac{\partial f}{\partial y} = 4xy - 3z^5 + 17x^2z$

$$\frac{\partial f}{\partial z} = -15yz^4 + 17x^2y$$

Ex. $f(x,y,z) = x^2\cos(2xy^3z+2y)$ find $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2)\cos(2xy^3z+2y) + x^2\frac{\partial}{\partial x}\left(\cos(2xy^3z+2y)\right)$$

$$= 2x\cos(2xy^3z+2y) + x^2\left(-\sin(2xy^3z+2y)\right)\cdot\frac{\partial}{\partial x}(2xy^3z+2y)$$

$$= 2x\cos(2xy^3z+2y) - x^2\sin(2xy^3z+2y)\cdot 2y^3z$$

$$= 2x\cos(2xy^3z+2y) - 2x^2y^3z\sin(2xy^3z+2y).$$

Implicit Differentiation:

Ex. $x^2+y^2+z^2 = 49$ find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(3,-2,6)$.

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) = \frac{\partial}{\partial x}(49)$$

y is constant, z is <u>not</u> constant, z depends on x & y.

$$2x + 0 + 2z\frac{\partial z}{\partial x} = 0$$

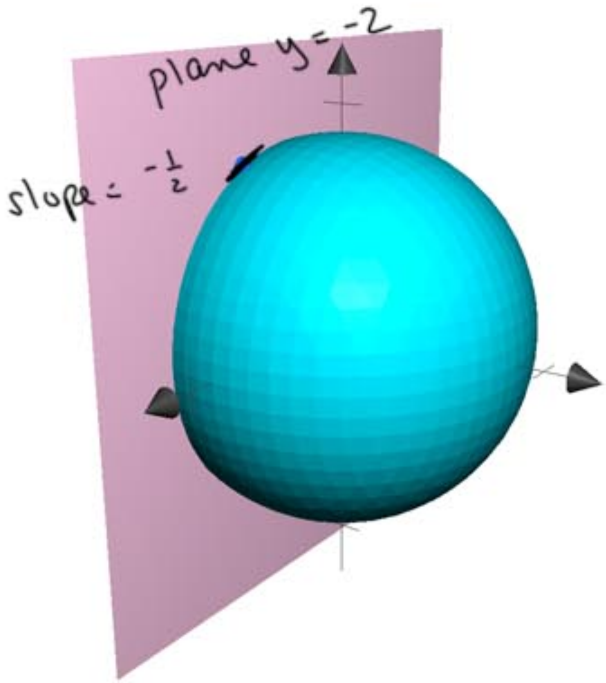
$$\frac{\partial z}{\partial x} = \frac{-2x}{2z} = -\frac{x}{z} \qquad \left.\frac{\partial z}{\partial x}\right|_{(3,-2,6)} = \frac{-3}{6} = -\frac{1}{2}.$$

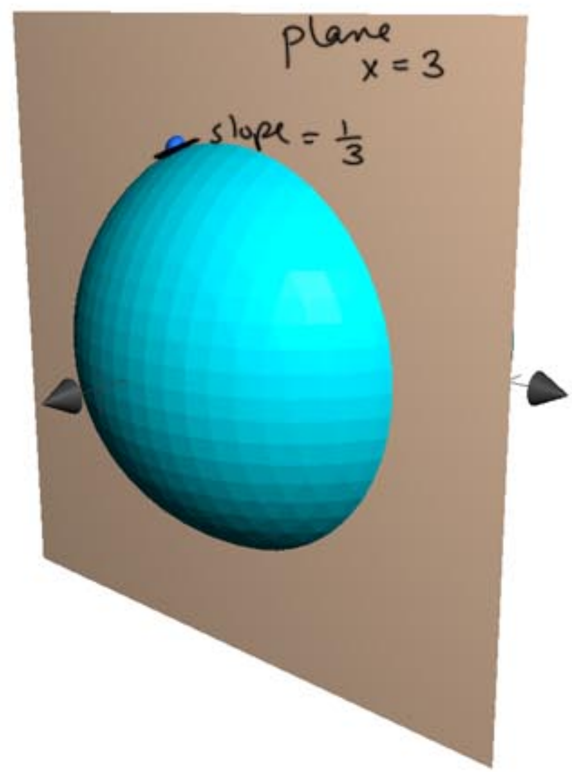
$$\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right) = \frac{\partial}{\partial y}(49)$$

x is constant,
z <u>not</u> constant

$$0 + 2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{2z} = -\frac{y}{z} \qquad \left.\frac{\partial z}{\partial y}\right|_{(3,-2,6)} = \frac{2}{6} = \frac{1}{3}.$$





Higher Order Partial Derivatives

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Ex. $f(x,y) = 2xy^2 - 5x^3y + 17x$

$f_x(x,y) = 2y^2 - 15x^2y + 17$ $\qquad$ $f_y(x,y) = 4xy - 5x^3$

$f_{xx}(x,y) = -30xy$ $\qquad$ $f_{yy}(x,y) = 4x$

$f_{xy}(x,y) = 4y - 15x^2$ $\qquad$ $f_{yx}(x,y) = 4y - 15x^2$

Ex. $f(x,y) = \arcsin(2xy^2)$

$$f_x(x,y) = \frac{1}{\sqrt{1-(2xy^2)^2}} \cdot \frac{\partial}{\partial x}\left(2xy^2\right) =$$

$$= \frac{1}{\sqrt{1-4x^2y^4}} \cdot 2y^2 = \frac{2y^2}{\sqrt{1-4x^2y^4}} = 2y^2\left(1-4x^2y^4\right)^{-1/2}$$

$$f_{xx}(x,y) = 2y^2 \cdot \left(-\tfrac{1}{2}\right)\left(1-4x^2y^4\right)^{-3/2} \frac{\partial}{\partial x}\left(1-4x^2y^4\right)$$

$$= \frac{-y^2}{\left(1-4x^2y^4\right)^{3/2}} \cdot \left(-8xy^4\right) = \frac{8xy^6}{\left(1-4x^2y^4\right)^{3/2}}$$

for $f_{xy}(x,y)$, $\quad f_x(x,y) = \dfrac{2y^2}{\sqrt{1-4x^2y^4}}$

$$f_{xy}(x,y) = \frac{\sqrt{1-4x^2y^4}\,\frac{\partial}{\partial y}\left(2y^2\right) - 2y^2\,\frac{\partial}{\partial y}\left(\sqrt{1-4x^2y^4}\right)}{1-4x^2y^4}$$

$$= \frac{\sqrt{1-4x^2y^4}\cdot(4y) - 2y^2\,\frac{1}{2}\left(1-4x^2y^4\right)^{-1/2}\overbrace{\frac{\partial}{\partial y}\left(1-4x^2y^4\right)}^{-16x^2y^3}}{1-4x^2y^4}$$

$$= \left(\frac{4y\sqrt{1-4x^2y^4} + 16x^2y^5\left(1-4x^2y^4\right)^{-1/2}}{1-4x^2y^4}\right)\frac{\sqrt{1-4x^2y^4}}{\sqrt{1-4x^2y^4}}$$

$$= \frac{4y(1-4x^2y^4)+16x^2y^5}{(1-4x^2y^4)^{3/2}} = \frac{4y-16x^2y^5+16x^2y^5}{(1-4x^2y^4)^{3/2}}$$

$$= \frac{4y}{(1-4x^2y^4)^{3/2}}$$

from $f(x,y) = \arcsin(2xy^2)$

$$f_y(x,y) = \frac{1}{\sqrt{1-4x^2y^4}} \cdot \frac{\partial}{\partial y}(2xy^2) = \frac{4xy}{\sqrt{1-4x^2y^4}}$$

$$f_{yy}(x,y) = \frac{\sqrt{1-4x^2y^4} \cdot 4x - 4xy \, \frac{1}{2}(1-4x^2y^4)^{-1/2}(-16x^2y^3)}{1-4x^2y^4}$$

$$= \frac{4x(1-4x^2y^4)+32x^3y^4}{(1-4x^2y^4)^{3/2}} = \frac{1+16x^3y^4}{(1-4x^2y^4)^{3/2}}$$

$$f_{yx}(x,y) = \frac{\sqrt{1-4x^2y^4}\;4y - 4xy\,\frac{1}{2}(1-4x^2y^4)^{-1/2}(-8xy^4)}{1-4x^2y^4}$$

$$= \frac{4y(1-4x^2y^4)+16x^3y^5}{(1-4x^2y^4)^{3/2}} = \frac{4y}{(1-4x^2y^4)^{3/2}} = f_{xy}$$

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Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$