Math 20300
Calculus III

Lesson 12
Limits

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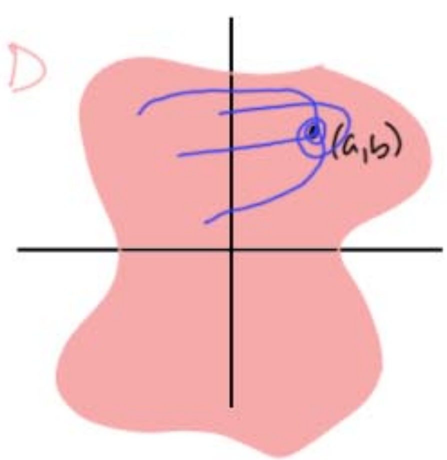
Limits

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

"the limit of f as (x,y) approaches (a,b) is L"

$(x,y) \to (a,b)$ means along any path in the domain D of f



with $y = f(x)$

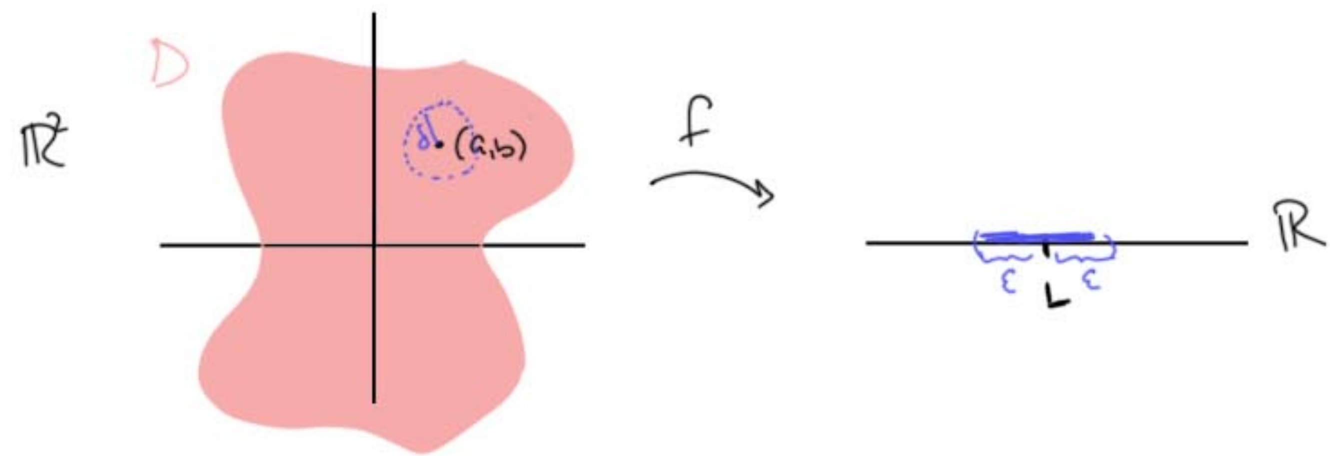
$$\lim_{x\to a^+} f(x) = \lim_{x \to a^-} f(x)$$

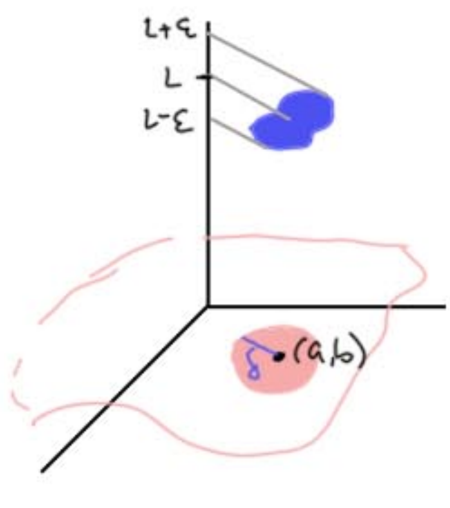
Epsilon-Delta Definition of the limit:

we say $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

if for any $\varepsilon > 0$, $\exists\ \delta > 0 \ni$ if $(x,y) \in D$ (domain)

and $\sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|f(x,y) - L| < \varepsilon$.



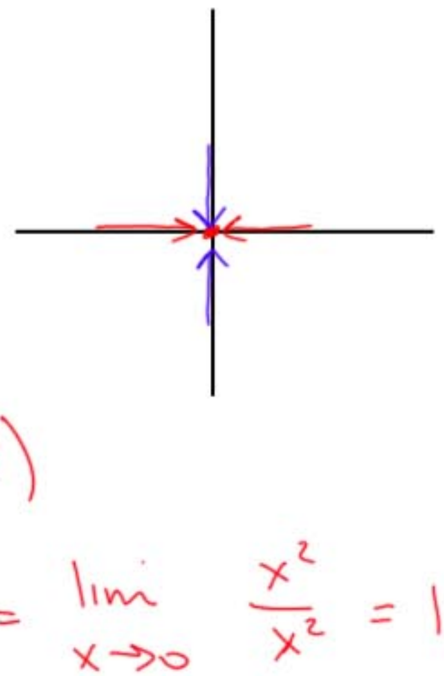


Find the limit or show that it does not exist.

Guess the limit, then show that for any $\varepsilon > 0$, $\exists\, \delta > 0$ $\ni \sqrt{(x-a)^2+(y-b)^2} < \delta$ $\Rightarrow |f(x,y) - L| < \varepsilon$.

find two paths along which the limits are different.

Ex. $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+2y^2}$$



limit along $y=0$ (horizontal)

$$\lim_{(x,0)\to(0,0)} \frac{x^2+0^2}{x^2+2(0)^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

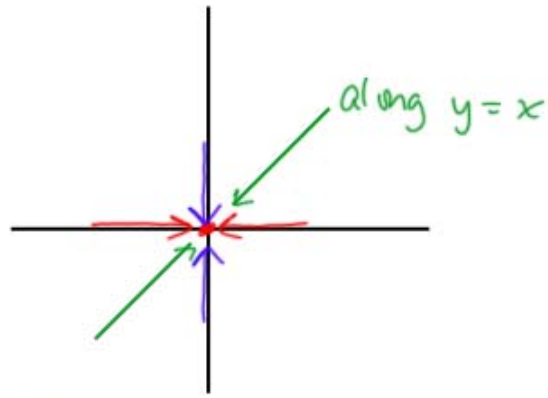
limit along $x=0$ (vertical)

$$\lim_{(0,y)\to(0,0)} \frac{0^2+y^2}{0^2+2y^2} = \lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2}$$

Since the limits are different along $\overbrace{x=0 \text{ and } y=0}^{\text{we found 2 paths along which limits } \neq}$,

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+2y^2} \text{ does not exist (DNE)}$$

Ex. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$



limit along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x(0)}{x^2+0^2} = 0.$$

limit along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{0(y)}{0^2+y^2} = 0.$$

limit along $y=x$ $\displaystyle\lim_{(x,x)\to(0,0)} \frac{x(x)}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

Since limit is different along $y=x$ and $y=0$,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ DNE}.$$

Ex. $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

limit along $y=0$ $\quad \lim_{(x,0)\to(0,0)} \frac{x(0)^2}{x^2+(0)^4} = 0$

limit along $x=0$ $\quad \lim_{(0,y)\to(0,0)} \frac{0\cdot y^2}{0^2+y^4} = 0$

limit along $y=x$ $\quad \lim_{(x,x)\to(0,0)} \frac{x\cdot \cancel{x^2}}{\underbrace{x^2+x^4}_{\cancel{x^2}(1+x^2)}} = 0$

limit along $x=y^2$ $\quad \lim_{(y^2,y)\to(0,0)} \frac{y^2\cdot y^2}{(y^2)^2+y^4}$

$$= \lim_{y\to 0} \frac{y^4}{\underbrace{y^4+y^4}_{2y^4}} = \frac{1}{2}$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ DNE

Ex. $\lim\limits_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$

limit along $y=0$ $\lim\limits_{(x,0)\to(0,0)} \frac{2x(0)^2}{x^2+0^2} = 0.$

as an exercise, examine limits along $x=0$
$y=x$

To prove the limit exists and $=0$, we need that
for every $\varepsilon>0$, $\exists\, \delta>0 \ni \sqrt{(x-0)^2+(y-0)^2} < \delta \Rightarrow$

$|f(x,y)-0| < \varepsilon.$

Start with $\varepsilon>0$, show that we can find such a δ.

$$\sqrt{x^2+y^2} < \delta \Rightarrow \left|\frac{2xy^2}{x^2+y^2}\right| < \varepsilon.$$

Let $\varepsilon>0$.

$$\left|\frac{2xy^2}{x^2+y^2}\right| = \frac{2|x|y^2}{x^2+y^2} \le 2|x|$$

$$\le 2\underbrace{\sqrt{x^2+y^2}} < \frac{\varepsilon}{2}$$

we know $y^2 \le x^2+y^2$

$\therefore \dfrac{y^2}{x^2+y^2} \le 1$

$|x| = \sqrt{x^2} \le \sqrt{x^2+y^2}$

let $\delta = \frac{\varepsilon}{2}$

$< 2 \cdot \frac{\varepsilon}{2} = \varepsilon$.

$\therefore$ The limit exists and $= 0$.

To show a limit exists without using the ε-δ definition, we can use <u>the Squeeze Theorem</u>:

Suppose that $|f(x,y) - L| \le g(x,y)$ $\forall (x,y)$ in some circle centered at (a,b), except possibly at (a,b).

If $\lim\limits_{(x,y)\to(a,b)} g(x,y) = 0$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

(next lesson)