Math 20300
Calculus III

Lesson 11
Functions of Several Variables

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Functions of Several Variables

in $\mathbb{R}^2$: $y = f(x)$ y is a function of x if for every x value in the domain there is one and only one corresponding y value

Ex. $y = x^2$ function $x^2 + y^2 = 1$ not a function

So far in $\mathbb{R}^3$, we've seen graphs of surfaces but haven't talked about whether or not they are graphs of functions in $\mathbb{R}^3$.

$z = f(x,y)$ is a function of two variables if for every pair (x,y) in the domain D, there is one and only one corresponding z-value.

note $D \subset \mathbb{R}^2$ here, $f: \mathbb{R}^2 \to \mathbb{R}$

$$z = f(x, y)$$

$\underbrace{(x,y)}_{}$ two input values

one output value

domain D is a region in $\mathbb{R}^2$,

range is a set in $\mathbb{R}$

Ex. $f(x,y) = 2x - 3y + 5$

notice $z = 2x - 3y + 5$

$2x - 3y - z + 5 = 0$ plane.

for $f(x,y) = 2x - 3y + 5$ domain: $\mathbb{R}^2$

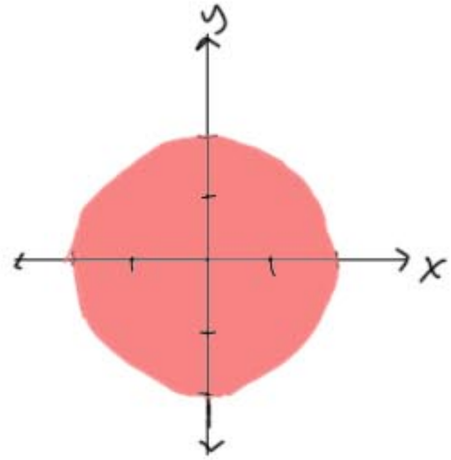
Ex. $f(x,y) = \sqrt{4 - x^2 - y^2}$

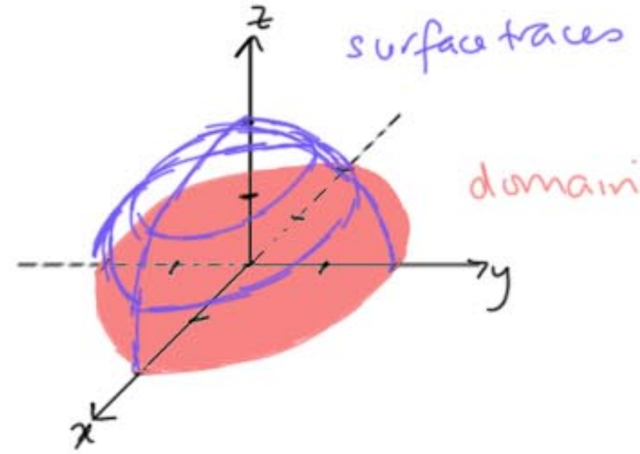
$z = \sqrt{4 - x^2 - y^2} \quad \leftarrow z \geq 0$

$z^2 = 4 - x^2 - y^2$ hemi-sphere radius 2

$x^2 + y^2 + z^2 = 4$ centered at the origin domain: $4-x^2-y^2 \geq 0$ $\qquad$ $4 \geq x^2+y^2$

$\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4\}$ $\qquad$ $\underbrace{x^2+y^2 \leq 4}$ circular disk



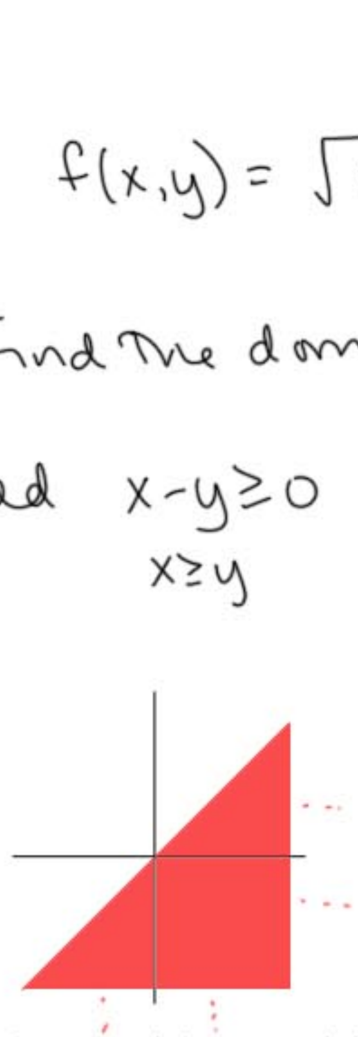


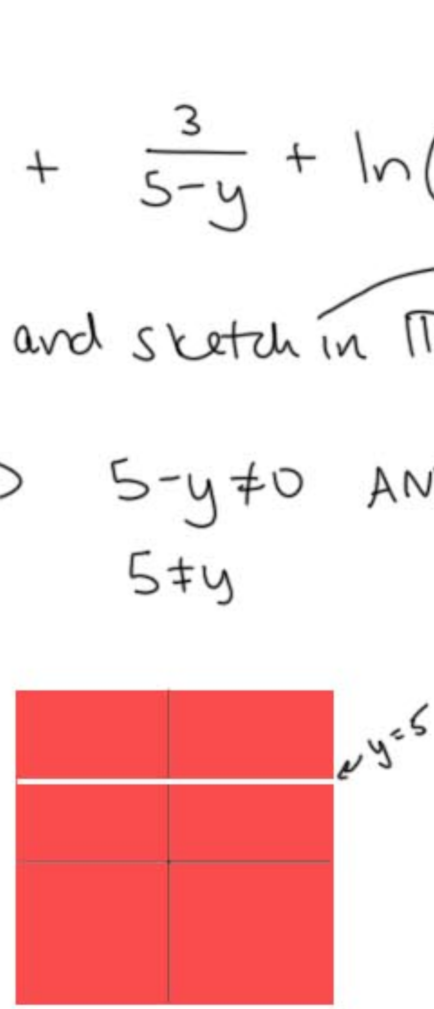
Ex. $f(x,y) = \sqrt{x-y} + \dfrac{3}{5-y} + \ln(x+15)$

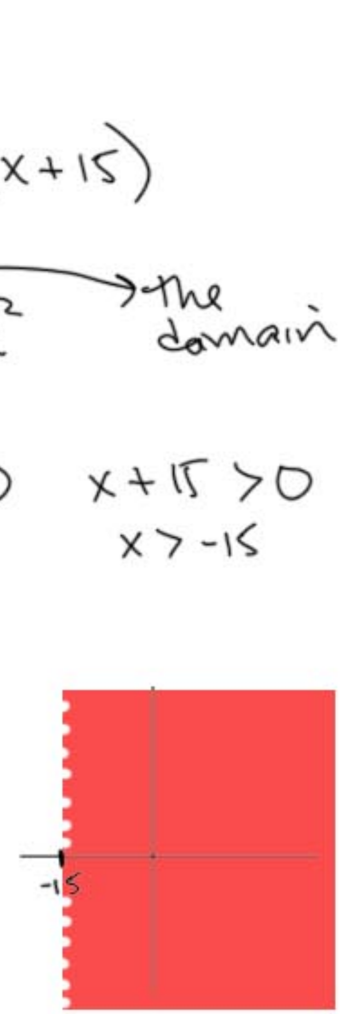
Find the domain and sketch in $\mathbb{R}^2$ the domain

need $x-y \geq 0$ AND $5-y \neq 0$ AND $x+15 > 0$

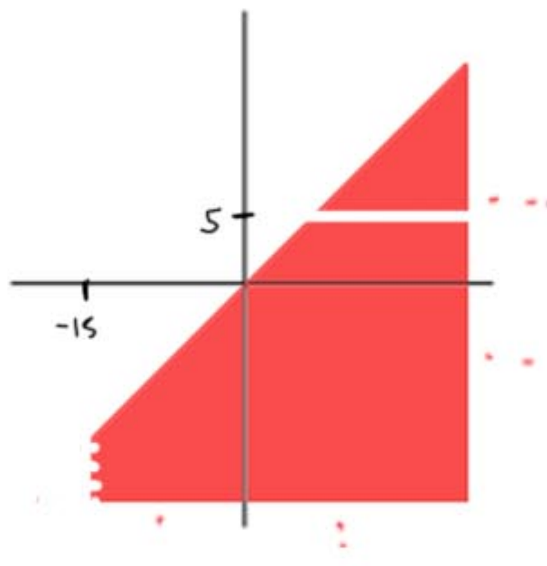
$x \geq y$ $\qquad$ $5 \neq y$ $\qquad$ $x > -15$







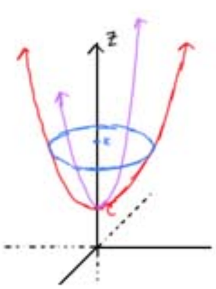
domain: $\{(x,y) \in \mathbb{R}^2 : x \geq y,\ y \neq 5, \text{ and } x > -15\}$



Graphing functions of two variables:

we already know how to graph functions that define quadric surfaces using traces.

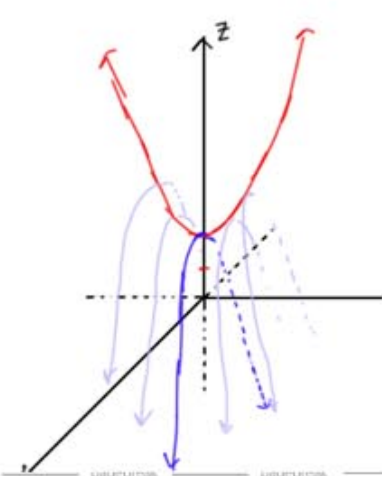
Ex. $f(x,y) = \sqrt{4 - x^2 - y^2}$ hemisphere



$g(x,y) = 2x^2 + y^2 + 4$ elliptic paraboloid

$h(x,y) = y^2 - x^2 + 2$

hyperbolic paraboloid



to graph any $z = f(x,y)$ surface,

we use <u>level curves</u> (traces of the form $z = \text{constant}$).

Ex. $\underbrace{f(x,y)}_{z} = e^{-xy}$

notice $z > 0$

start with $z = 1$

$$1 = e^{-xy}$$

$$\ln 1 = -xy$$

$$0 = xy$$

$x = 0$, y any value

or

$y = 0$ x any value

$z = 2$

$$2 = e^{-xy}$$

$$xy = -\ln 2$$

$$y = \frac{-\ln 2}{x}$$

$y = -\frac{1}{x}$

$z = 3$

$3 = e^{-xy} \qquad xy = -\ln 3 \qquad y = \dfrac{-\ln 3}{x}$

$z = 1/2 \qquad 1/2 = e^{-xy}$

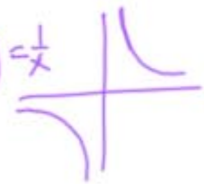
$$\ln\left(\tfrac{1}{2}\right) = -xy$$

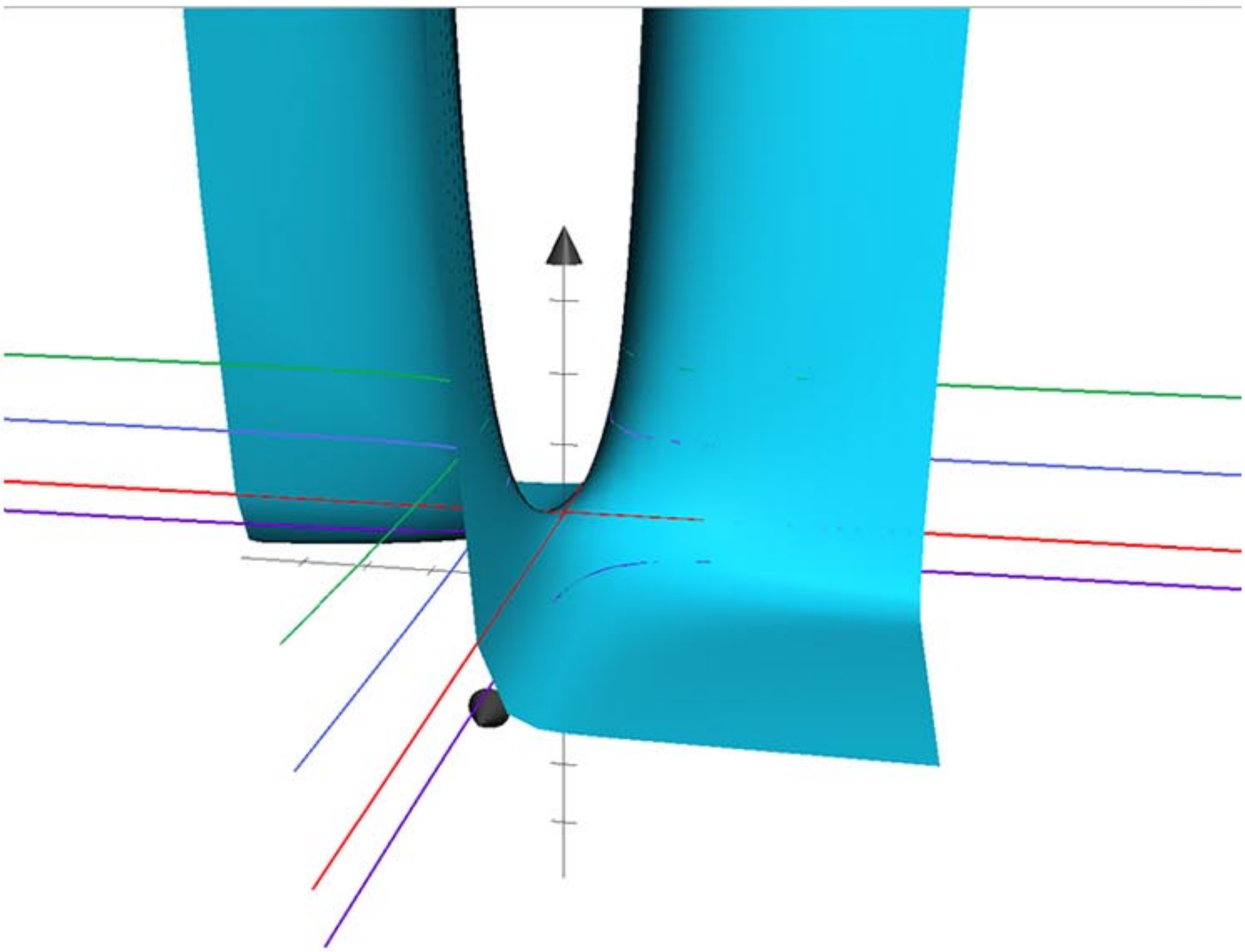
$$-\ln 2 = -xy$$

$$xy = \ln 2$$

$$y = \frac{\ln 2}{x}$$

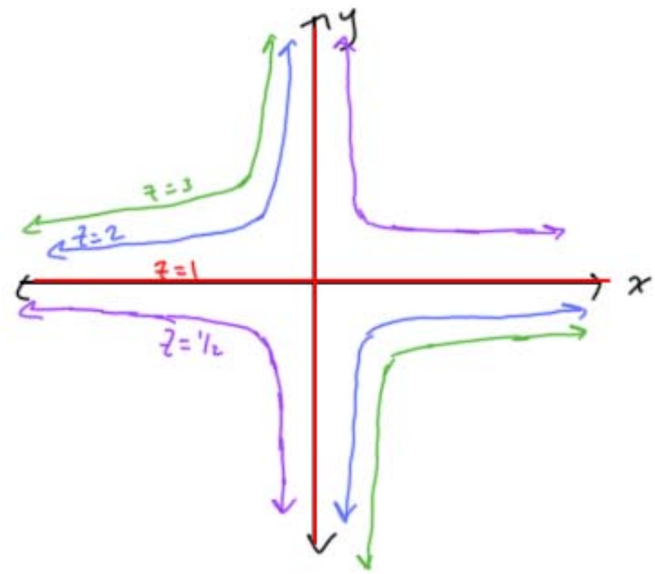
$y = \frac{1}{x}$



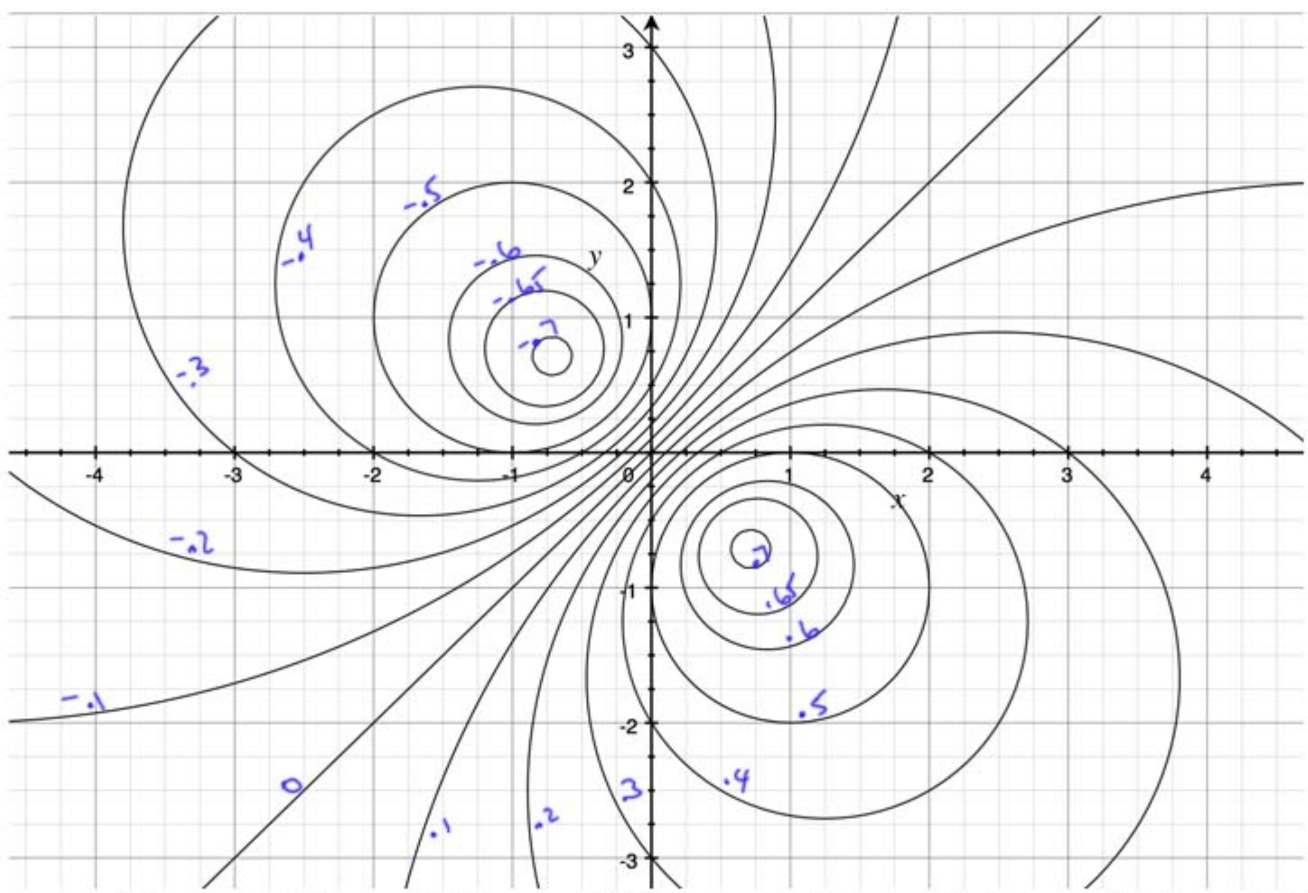


we can also consider the <u>contour plot</u>

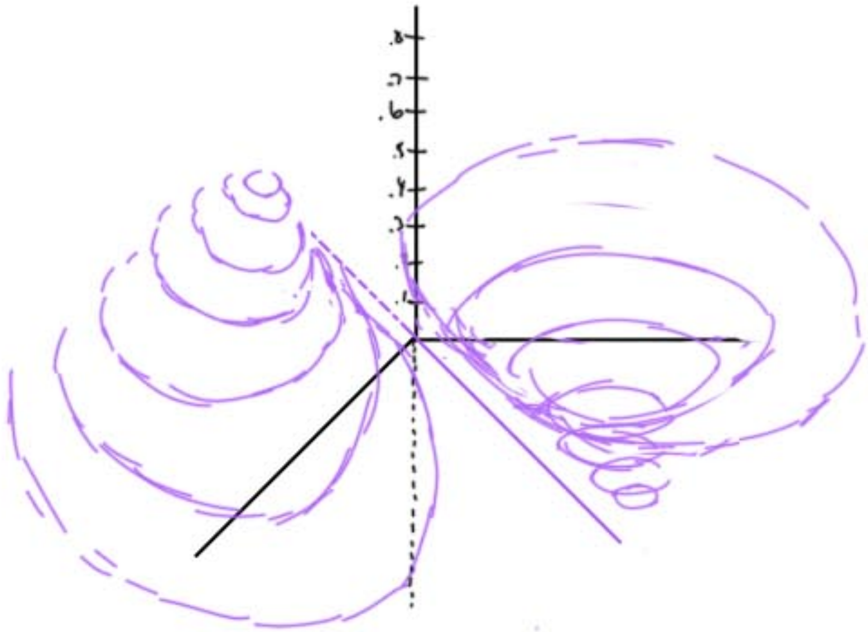
a 2D plot of the level curves:

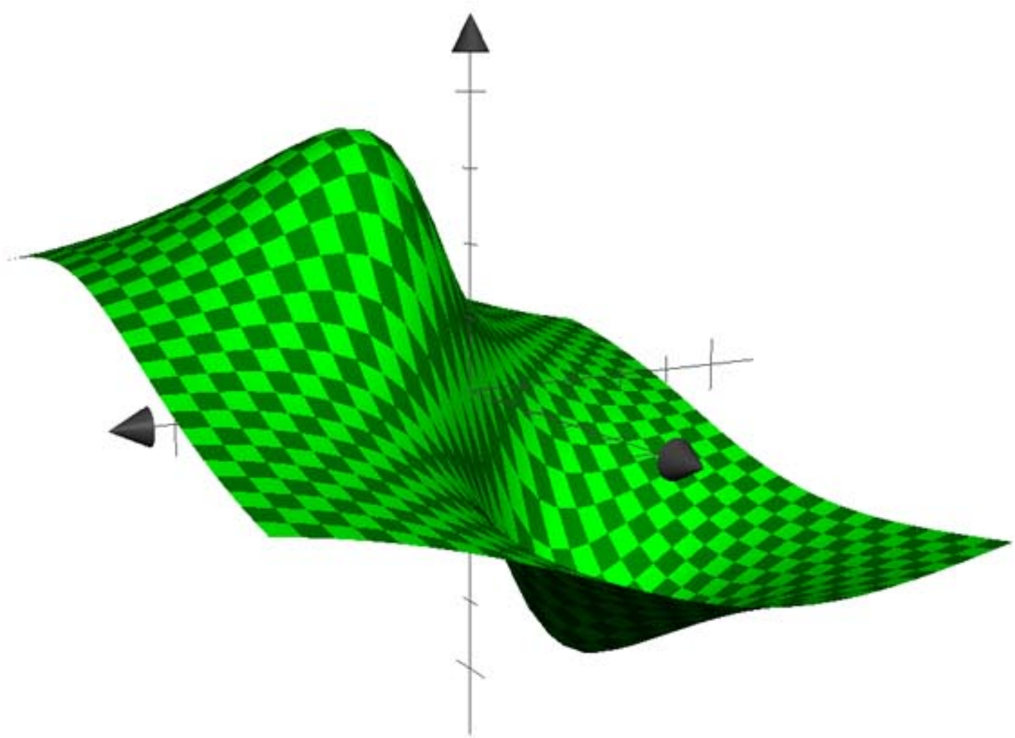


Ex.

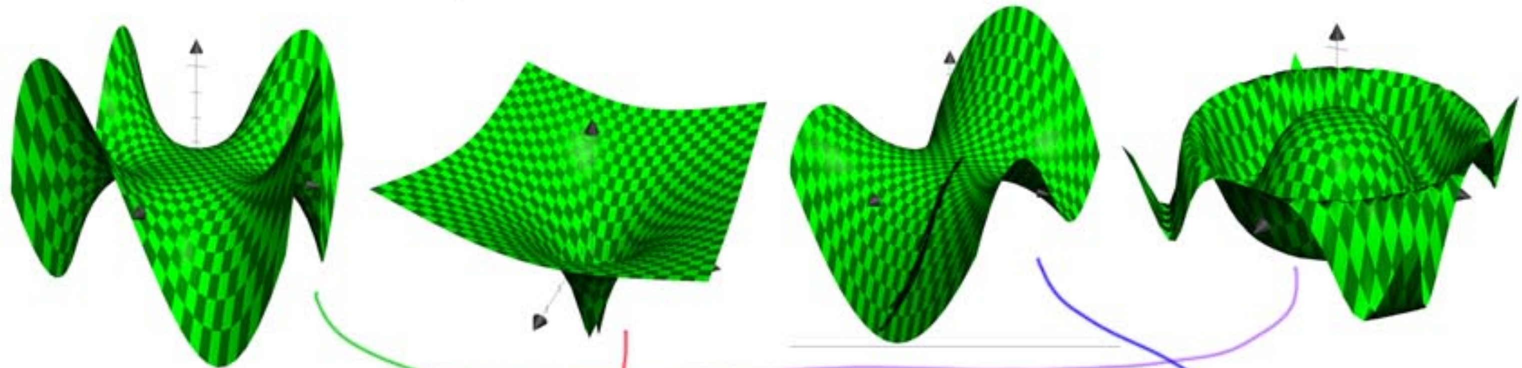


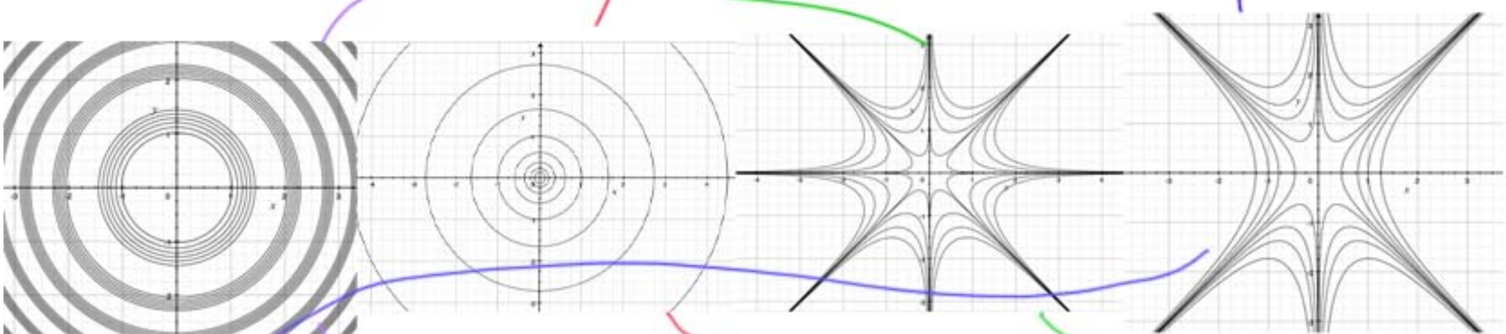
Given the above contour map, sketch the surface in $\mathbb{R}^3$





Ex. match the contour plots with their equations and surfaces.





a) $z = xy^2 - x^3$

$0 = xy^2 - x^3$

$0 = x(y^2 - x^2)$

$x = 0$

$y = \pm x$

b) $z = \cos(x^2+y^2)$

$1 = \cos(x^2+y^2)$

$x^2+y^2 = 0, 2\pi, 4\pi, \ldots$

c) $z = \ln(x^2+y^2)$

$1 = \ln(x^2+y^2)$

$e = x^2+y^2$

circle radius $\sqrt{e}$

d) $z = xy^3 - yx^3$

$0 = xy^3 - yx^3$

$0 = xy(y^2 - x^2)$

$x = 0$

$y = 0$

$y = \pm x$

Functions of three variables

$$\underbrace{w}_{\substack{\text{one}\\\text{output}\\\text{variable}}} = f(\underbrace{x,y,z}_{\substack{\text{three}\\\text{input}\\\text{variables}}})$$

$f : \mathbb{R}^3 \to \mathbb{R}$

domain is a subset of $\mathbb{R}^3$

can't graph in 4 dimensions, but we can get some useful information looking at the domain and using level surfaces (the three dimensional version of level curves)

Ex. $w = f(x,y,z)$ could model the temperature in a room.

could be warmer near the door,

cooler near the air conditioner,

so temperature would depend on the point in space in the room.

Ex. $f(x,y,z) = \tan\left(\frac{x}{10}\right) + \ln\left(x^2+y^2-4\right) + \frac{2}{4-z^2}$

Domain: $\frac{x}{10} \neq (2k+1)\frac{\pi}{2} \quad k \in \mathbb{Z}$

$x^2+y^2-4>0$

$4-z^2 \neq 0$

$\frac{x}{10} \neq (2k+1)\frac{\pi}{2}$

$x \neq 5(2k+1)\pi$

$x \neq \pm 5\pi$
$\pm 15\pi$
$\pm 25\pi \ldots$

$x^2+y^2-4>0$

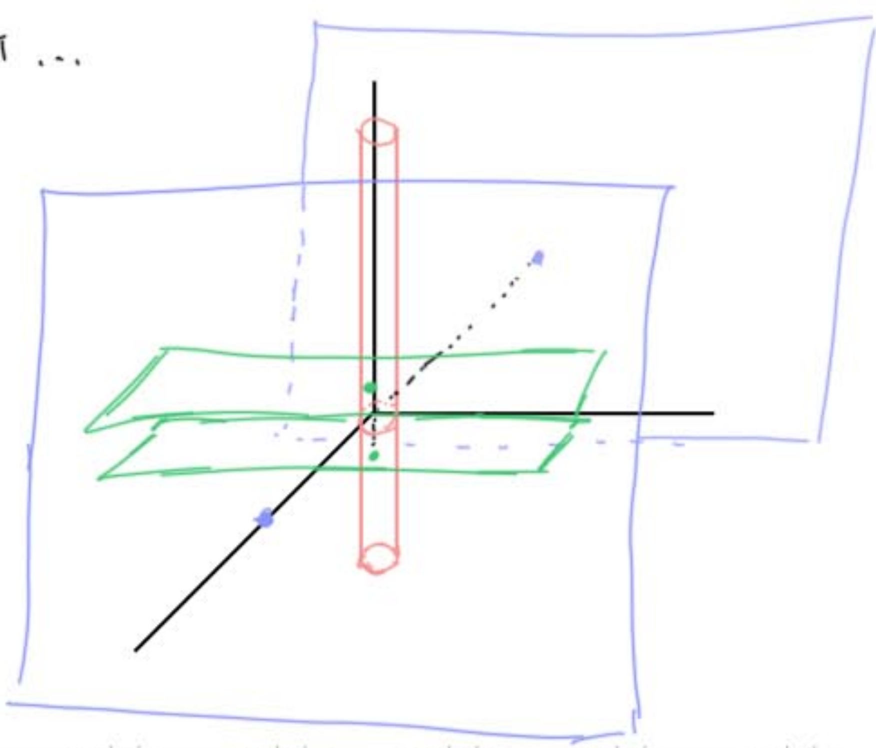
$x^2+y^2>4$

outside the cylinder

$4-z^2 \neq 0$

$z^2 \neq 4$

$z = \pm 2$



Level surfaces

for functions $w = f(x,y,z)$ we can consider the surfaces in $\mathbb{R}^3$ given by $w =$ constant.

Ex. $\underbrace{f(x,y,z)}_{\text{set = constant}} = x^2+y^2+z^2$

$1 = x^2+y^2+z^2$

$4 = x^2+y^2+z^2$ spheres of different radii

$9 = x^2+y^2+z^2$

$0 = x^2+y^2+z^2$ just the point $(0,0,0)$