Math 20300
Calculus III

Lesson 6
Planes

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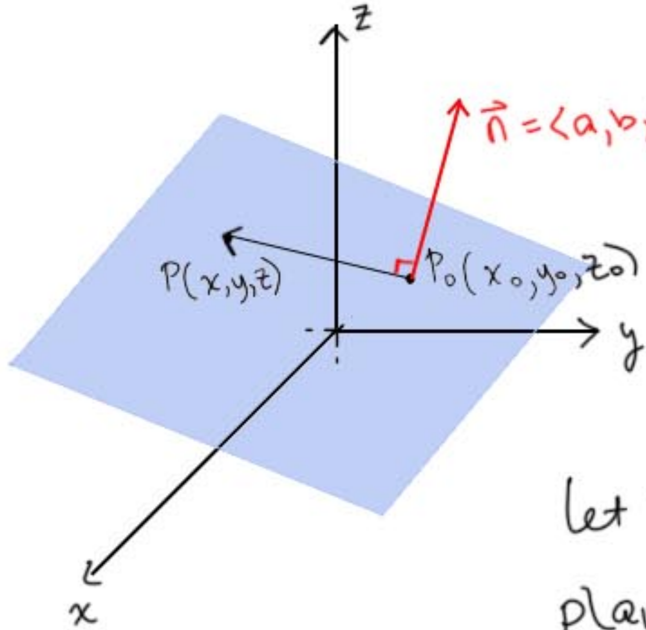
Planes

Planes are specified by a <u>point</u> on the plane and a <u>normal vector</u> to the plane

↓

vector that is orthogonal to all vectors contained in the plane.



let P_0 be a point on the plane with normal vector $\vec{n} = \langle a,b,c \rangle$, then

for any point $P(x,y,z)$ on the plane, vector $\overrightarrow{P_0P}$ is orthogonal to vector $\vec{n}$.

$$\overrightarrow{P_0P} = \langle x-x_0, y-y_0, z-z_0 \rangle \qquad \vec{n} = \langle a,b,c \rangle$$

$$+ \quad \overrightarrow{P_0P} \cdot \vec{n} = 0$$

$$\boxed{\begin{array}{l} a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \\ \text{Equation of the plane through } (x_0,y_0,z_0) \\ \text{with normal vector } \langle a,b,c\rangle. \end{array}}$$

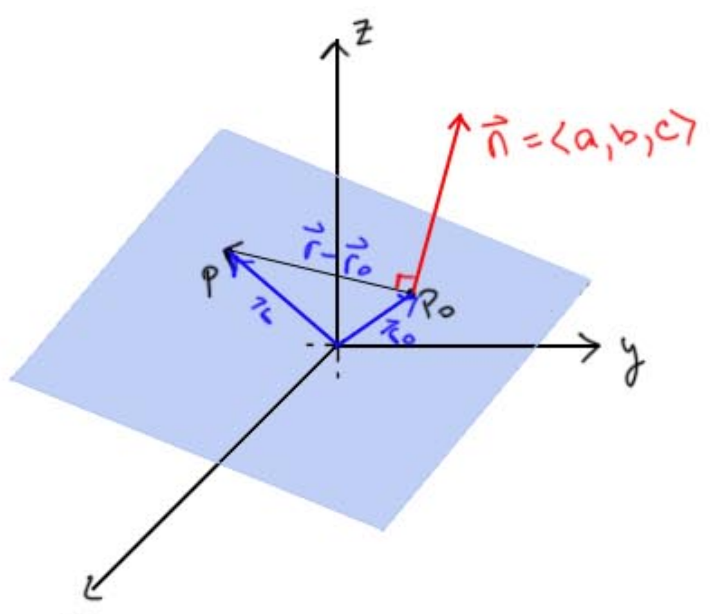
If we multiply out, we'll get

$$ax+by+cz+(-ax_0-by_0-cz_0)=0$$

$$ax+by+cz+d=0.$$

note that we can still read off normal vector $\langle a,b,c\rangle$.

In vector form,



$$\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$$

$$\vec{n}\cdot\vec{r}-\vec{n}\cdot\vec{r}_0=0$$

$$\vec{n}\cdot\vec{r}=\vec{n}\cdot\vec{r}_0$$

vector equation of the plane.

Ex. Find an equation of the plane perpendicular to $\langle 3,-2,5\rangle$ and passing through $(2,-4,-3)$.

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$3(x-2)-2(y+4)+5(z+3)=0$$

$$3x-6-2y-8+5z+15=0$$

$$3x-2y+5z+1=0.$$

Ex. From lesson 4, we had:

Find a vector normal to $\left(\begin{array}{c}\text{orthogonal to}\\ \text{perpendicular to}\end{array}\right)$ the plane containing the points $P(3,0,4)$, $Q(1,0,1)$, $R(2,3,0)$

we found vectors

$\overrightarrow{QP}=\langle 2,0,3\rangle$ & $\overrightarrow{QR}=\langle 1,3,-1\rangle$

$\vec{QP} \times \vec{QR} = -9\vec{i} + 5\vec{j} + 6\vec{k}$ normal to the plane.

$= \langle -9, 5, 6 \rangle$

Now, let's find an equation of the plane.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$\langle a, b, c \rangle = \langle -9, 5, 6 \rangle$ but we can choose P, Q, or R for (x_0, y_0, z_0).

Choosing $P(3, 0, 4)$:

$$-9(x-3) + 5(y-0) + 6(z-4) = 0$$

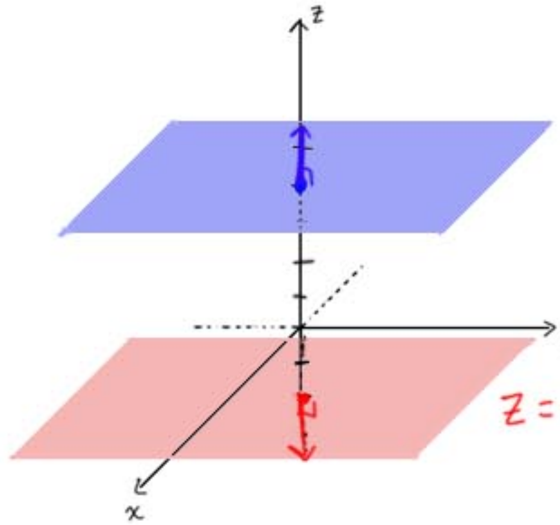
$$-9x + 27 + 5y + 6z - 24$$

$$-9x + 5y + 6z + 3 = 0$$

Choosing Q or R gives same simplified form of the equation of this plane. (exercise).

Parallel Planes

Two planes are said to be parallel if their normal vectors are parallel.



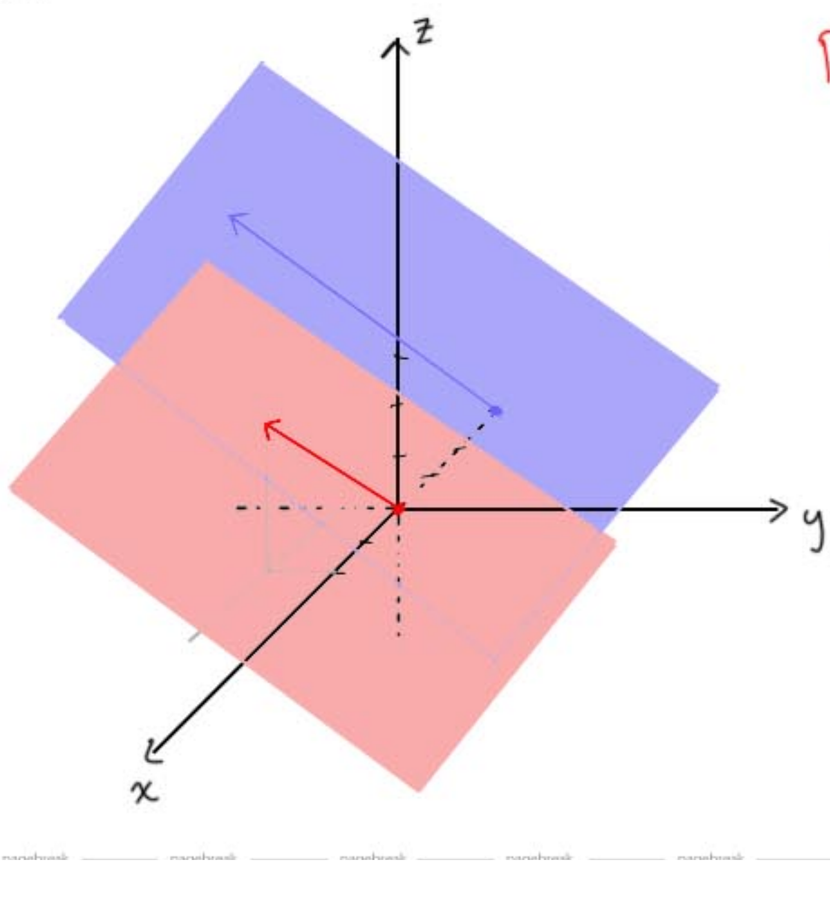
$z = 4$ $(0,0,4)$

$\langle 0,0,1 \rangle$

$z = -2$ $(0,0,-2)$

$\langle 0,0,-1 \rangle$

Ex.



plane #1: $(0,0,0)$

$\langle 2,-1,3 \rangle$

plane #2: $(-3,0,0)$

$\langle 4,-2,6 \rangle$

Intersecting Planes

If two planes are not parallel, they are intersecting planes. Their intersection is a line. The angle between the planes (dihedral angle) is the acute angle between their normal vectors.

Ex. Do the planes intersect? If so, find the dihedral angle.

Plane #1:

$$x + 3y + 2z - 6 = 0$$

$$\vec{n}_1 = \langle 1, 3, 2 \rangle$$

Plane #2

$$x - y + 2z - 2 = 0$$

$$\vec{n}_2 = \langle 1, -1, 2 \rangle$$

not parallel.

$\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\| \, \|\vec{n}_2\| \cos\theta$ where θ is the angle between $\vec{n}_1$ + $\vec{n}_2$

$$\|\vec{n}_1\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

$$\|\vec{n}_2\| = \sqrt{1^2+(-1)^2+2^2} = \sqrt{6}$$

$$\vec{n}_1 \cdot \vec{n}_2 = 1(1) + 3(-1) + 2(2) = 1-3+4 = 2$$

$$\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\| \|\vec{n}_2\| \cos\theta$$

$$2 = \sqrt{14}\sqrt{6} \cos\theta$$

$$\cos\theta = \frac{2}{\sqrt{14}\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{14}\sqrt{6}}\right) \approx 1.35 \text{ rad}$$

$\approx 77^\circ$ ← is acute.

NOTE
if we had $\vec{n}_2 = \langle -1, 1, -2 \rangle$ for example, we'd have

$$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{14}\sqrt{6}}\right) \approx 103^\circ$$ ← not acute, so take supplement.

Ex. Find an equation of the plane containing line $x = 2+3t$, $y = t$, $z = 1-2t$ and point $(2,1,1)$

$$x = 2+3t$$
$$y = t$$
$$z = 1-2t$$

equation of plane $\quad a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

we have $(x_0, y_0, z_0) = (2,1,1)$

we need normal vector to the plane $\vec{n} = \langle a,b,c \rangle$.

from the given point and line, we need a normal vector.

how can we get a vector that is orthogonal to other vectors? cross product.

we need two vectors (that are not parallel) in the plane, and take their cross product.

Work on this problem
on your own

given point $(2,1,1)$ P

on the line $(2,0,1)$ Q $\quad$ (when $t=0$)

$(5,1,-1)$ R $\quad$ (when $t=1$)

$\overrightarrow{QP} = \langle 0, 1, 0\rangle = \vec{j}$ $\qquad \overrightarrow{QR} = \langle 3, 1, -2\rangle$

$$\langle 0,1,0\rangle \times \langle 3,1,-2\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 3 & 1 & -2 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix}\vec{i} - \begin{vmatrix} 0 & 0 \\ 3 & -2 \end{vmatrix}\vec{j} + \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}\vec{k} =$$

$$(-2-0)\vec{i} - (0-0)\vec{j} + (0-3)\vec{k} =$$

$$-2\vec{i} - 3\vec{k} = \langle -2, 0, -3\rangle$$ normal to plane

notice also $\langle 2, 0, 3\rangle$ normal to The plane

given point $(2,1,1)$

equation of plane

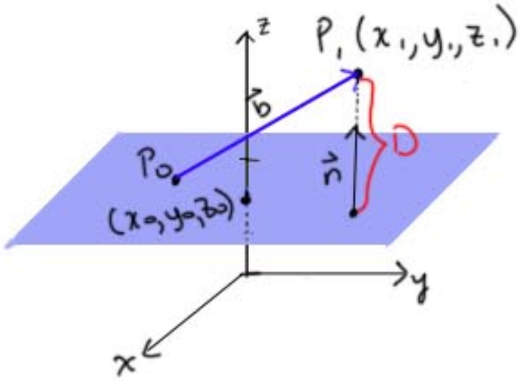
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$2(x-2) + 0(y-1) + 3(z-1) = 0$$

$$2x - 4 + 3z - 3 = 0$$

$$2x + 3z - 7 = 0.$$

Distance from a point to a plane



Plane: $ax + by + cz + d = 0$

$\vec{n} = \langle a, b, c \rangle$

$\vec{b} = \langle x_1 - x_0,\, y_1 - y_0,\, z_1 - z_0 \rangle$

$$D = \left|\mathrm{comp}_{\vec{n}}\, \vec{b}\right| = \frac{|\vec{n}\cdot\vec{b}|}{\|\vec{n}\|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Ex. Find The distance from point $(1, 0, 5)$ to plane $2x + 3y - z + 6 = 0$

$$D = \frac{|2(1) + 3(0) - 1(5) + 6|}{\sqrt{2^2 + 3^2 + (-1)^2}} = \frac{3}{\sqrt{14}}.$$

Ex. Where does the line through $(1,0,1)$ and $(-3,0,2)$ intersect the plane

$$x+2y+z+5=0 ?$$

line through $(1,0,1)$ and $(-3,0,2)$ has direction $\langle 4,0,-1\rangle$

so equations: $x = 1+4t$

$y = 0$

$z = 1-t$

if the line intersects the plane, $\exists\, t \ni$ (there exists ... such that)

$$(1+4t)+2(0)+(1-t)+5=0$$

so $1+4t+1-t+5=0$

$3t+7=0$

$t=-7/3$

$x = 1+4(-7/3)$

$= 1-28/3 = -\frac{25}{3}$

$y=0$ $\quad z = 1-(-7/3)$

$1+7/3 = 10/3$

$$\left(-\frac{25}{3}, 0, \frac{10}{3}\right)$$