Math 20300
Calculus III

Lesson 4
The Cross Product

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The Cross Product

first some definitions :

the <u>determinant of a 2x2 matrix</u> (array) of real numbers :

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$$

Ex. $$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 3(2) = -2$$

the determinant of a 3x3 matrix of real numbers:

$$\begin{vmatrix} \boxed{a_1} & \boxed{a_2} & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underline{a_1} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \underline{a_2} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Ex. $$\begin{vmatrix} 1 & 0 & 3 \\ 2 & -1 & 5 \\ 4 & -2 & 4 \end{vmatrix} = 1 \begin{vmatrix} -1 & 5 \\ -2 & 4 \end{vmatrix} - 0 \begin{vmatrix} 2 & 5 \\ 4 & 4 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$$

$$= 1\left((-1)(4) - (-2)(5)\right) - 0\left(2(4) - 4(5)\right) + 3\left(2(-2) - 4(-1)\right)$$

$$= (-4+10) - 0 + 3(-4+4) = 6.$$

<u>The cross product</u> of $\vec{a}$ and $\vec{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where
$\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

vector product
(result is a <u>vector</u>)

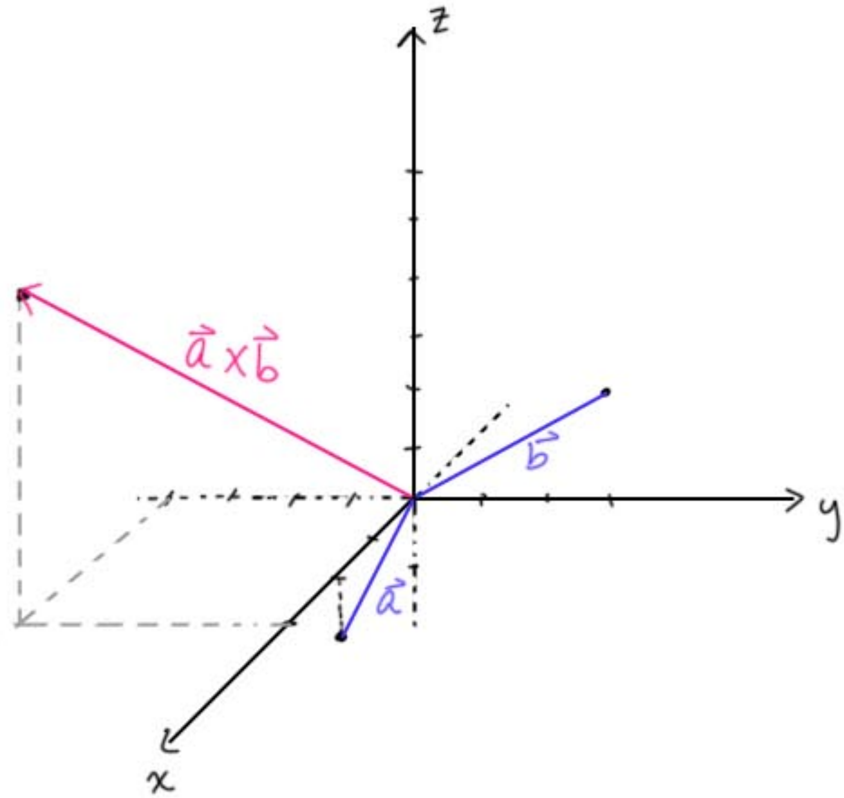
Ex. $\vec{a} = \langle 2, 0, -1 \rangle \quad \vec{b} = \langle 0, 3, 2 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 0 & 3 & 2 \end{vmatrix} =$$

$$\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \vec{k} =$$

$$\left(0(2) - 3(-1)\right)\vec{i} - \left(2(2) - 0(-1)\right)\vec{j} + \left(2(3) - 0(0)\right)\vec{k} =$$

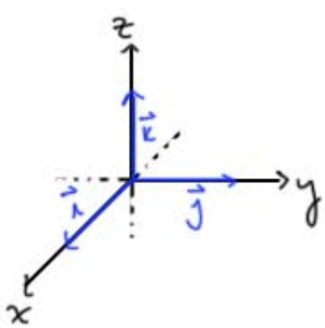
$$3\vec{i} - 4\vec{j} + 6\vec{k} = \langle 3, -4, 6 \rangle$$ vector result.



Ex. $\underset{\langle 1,0,0\rangle}{\vec i} \times \underset{\langle 0,1,0\rangle}{\vec j} = \begin{array}{r} \\ \vec i \to \\ \vec j \to \end{array}\begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} =$

$$= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \vec i - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \vec j + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec k =$$

$$= 0\vec i - 0\vec j + 1\vec k = \vec k.$$



Theorem: Vector $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and orthogonal to $\vec{b}$.

Proof:

from lesson 3, $\vec{a}$ & $\vec{b}$ are orthogonal iff $\vec{a} \cdot \vec{b} = 0$

So we need to show that $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

and $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= (a_2 b_3 - b_2 a_3)\vec{i} - (a_1 b_3 - b_1 a_3)\vec{j} + (a_1 b_2 - b_1 a_2)\vec{k}$$

$$\therefore (\vec{a} \times \vec{b}) \cdot \vec{a} = (a_2 b_3 - b_2 a_3) a_1 - (a_1 b_3 - b_1 a_3) a_2 + (a_1 b_2 - b_1 a_2) a_3$$

$$= a_2 \cancel{b_3} a_1 - b_2 \cancel{a_3} a_1 - a_1 \cancel{b_3} a_2 + b_1 a_3 a_2 + a_1 \cancel{b_2} a_3 - b_1 a_2 a_3$$

$$= 0.$$

a similar computation shows $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$ (exercise) $\equiv$

Theorem: If θ is the angle between $\vec{a}$ & $\vec{b}$ (with $0 \le \theta \le \pi$) then $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$

Proof: we'll prove $\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2\theta$

then since $\sin\theta \ge 0$ for $0 \le \theta \le \pi$,

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta.$$

$\vec{a} \times \vec{b} =$

$$(a_2 b_3 - b_2 a_3)\vec{i} - (a_1 b_3 - b_1 a_3)\vec{j} + (a_1 b_2 - b_1 a_2)\vec{k}$$

so, $\|\vec{a} \times \vec{b}\|^2 = (a_2 b_3 - b_2 a_3)^2 + (a_1 b_3 - b_1 a_3)^2 + (a_1 b_2 - b_1 a_2)^2$

$$= a_2^2 b_3^2 - 2a_2 b_3 b_2 a_3 + b_2^2 a_3^2 + a_1^2 b_3^2 - 2a_1 b_3 b_1 a_3 + b_1^2 a_3^2 + a_1^2 b_2^2 - 2a_1 b_2 b_1 a_2 + b_1^2 a_2^2$$

add in and subtract $+ a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 - a_1^2 b_1^2 - a_2^2 b_2^2 - a_3^2 b_3^2 =$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 =$$

$$= \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$$

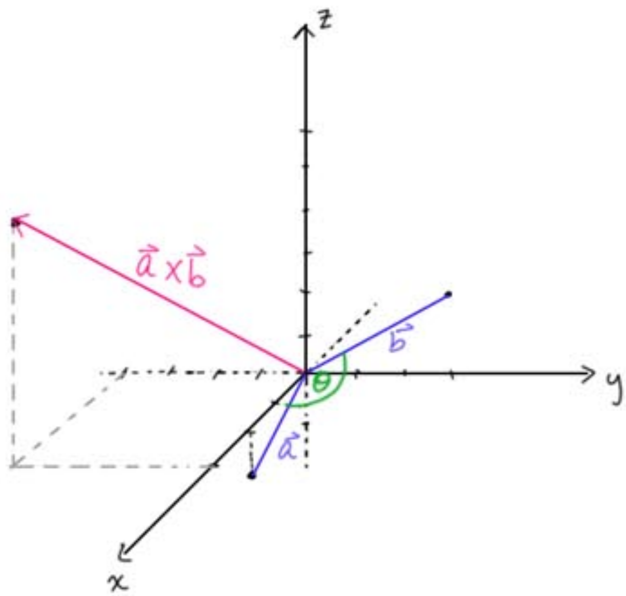
$$= \|\vec{a}\|^2 \|\vec{b}\|^2 - \|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2\theta$$ since $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

$$= \|\vec{a}\|^2 \|\vec{b}\|^2 (1-\cos^2\theta)$$

$$= \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2\theta. \quad //$$

Ex. In above example,

$\vec{a} = \langle 2, 0, -1 \rangle \quad \vec{b} = \langle 0, 3, 2 \rangle$



$\vec{a}\times\vec{b} = \langle 3, -4, 6 \rangle$

we can find $\measuredangle\,\theta$.

$\|\vec{a}\| = \sqrt{5} \quad \|\vec{b}\| = \sqrt{13}$

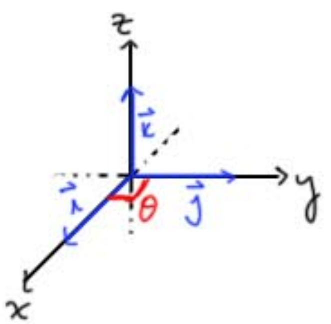
$\|\vec{a}\times\vec{b}\| = \sqrt{61}$

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$$

$$\sqrt{61} = \sqrt{5}\sqrt{13}\sin\theta$$

$$\sin\theta = \frac{\sqrt{61}}{\sqrt{5}\sqrt{13}} \qquad \theta = \sin^{-1}\left(\frac{\sqrt{61}}{\sqrt{5}\sqrt{13}}\right) \approx \boxed{\begin{array}{l} 1.32 \text{ rad} \\ \approx 75.6^\circ \end{array}}$$

Ex. above $\vec{i} \times \vec{j} = \vec{k}$ we know $\theta = \frac{\pi}{2}$



$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$$

$$\|\vec{i} \times \vec{j}\| \stackrel{?}{=} \|\vec{i}\| \, \|\vec{j}\| \sin\left(\tfrac{\pi}{2}\right)$$

$$\|\vec{k}\| \stackrel{?}{=} \|\vec{i}\| \, \|\vec{j}\| \sin\left(\tfrac{\pi}{2}\right)$$

$$1 = 1 \cdot 1 \cdot 1 \checkmark$$

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* right hand rule

curl fingers of right hand from $\vec{a}$ to $\vec{b}$, then thumb gives $\vec{a} \times \vec{b}$

<u>Corollary</u>: Two non-zero vectors $\vec{a}$ + $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

<u>Proof</u>: recall, vectors are parallel if they are scalar multiples of one another.

This means the angle between them is 0 or π.

$\therefore$ if $\vec{a}$ + $\vec{b}$ are parallel, $\sin\theta = 0$

$$\therefore \|\vec{a} \times \vec{b}\| = 0, \Rightarrow \vec{a} \times \vec{b} = \vec{0}.$$

Conversely, if $\vec{a} \times \vec{b} = \vec{0}$, then $\|\vec{a} \times \vec{b}\| = 0$

$$\text{and} \quad 0 = \underbrace{\|\vec{a}\| \, \|\vec{b}\|}_{\neq 0} \sin\theta$$

$$\therefore \sin\theta = 0, \text{ and } \theta = 0 \text{ or } \theta = \pi$$

$\therefore \vec{a} + \vec{b}$ are parallel.

Properties of the cross product:

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$ c scalar
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
5. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Proof of 5 (others left as an exercise)

$\vec{a} \times (\vec{b} \times \vec{c}) =$

$\langle a_1, a_2, a_3 \rangle \times \langle b_2c_3 - c_2b_3, \, c_1b_3 - b_1c_3, \, b_1c_2 - c_1b_2 \rangle$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - c_2b_3 & c_1b_3 - b_1c_3 & b_1c_2 - c_1b_2 \end{vmatrix}$$

$$= \left(a_2(b_1c_2 - c_1b_2) - a_3(c_1b_3 - b_1c_3) \right) \vec{i} \quad +$$

$$- \left(a_1(b_1c_2 - c_1b_2) - a_3(b_2c_3 - c_2b_3) \right) \vec{j} \quad +$$

$$+ \left(a_1(c_1b_3 - b_1c_3) - a_2(b_2c_3 - c_2b_3) \right) \vec{k}$$

$$= \left(a_2b_1c_2 - a_2c_1b_2 - a_3c_1b_3 + a_3b_1c_3 \right) \vec{i} \quad +$$

$$+ \left(-a_1b_1c_2 + a_1c_1b_2 + a_3b_2c_3 - a_3c_2b_3 \right) \vec{j}$$

$$+ \left(a_1c_1b_3 - a_1b_1c_3 - a_2b_2c_3 + a_2c_2b_3 \right) \vec{k}$$

$\vec{b}$

$\vec{c}$

$$= (a_2c_2 + a_3c_3)b_1 \vec{i} \quad - (a_2b_2 + a_3b_3)c_1 \vec{i} \quad +$$

$$+ a_1b_1c_1 \vec{i} \qquad - a_1b_1c_1 \vec{i}$$

$$+ (a_1c_1 + a_3c_3)b_2 \vec{j} - (a_1b_1 + a_3b_3)c_2 \vec{j} \quad +$$

$$+ a_2b_2c_2 \vec{j} \qquad - a_2b_2c_2 \vec{j}$$

$$+\,(a_1c_1+a_2c_2)b_3\vec{k} - (a_1b_1+a_2b_2)c_3\vec{k}$$

$$+\,a_3b_3c_3\vec{k} - a_3b_3c_3\vec{k}$$

$$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}.$$

Ex. Find a vector normal to $\left(\begin{array}{c}\text{orthogonal to}\\ \text{perpendicular to}\end{array}\right)$ the plane containing the points $(3,0,4)$, $(1,0,1)$, $(2,3,0)$.

We know how to find a vector orthogonal to to other vectors, so...

lets create two vectors from our points!

Work on this problem on your own

$P(3,0,4) \quad Q(1,0,1) \quad R(2,3,0)$

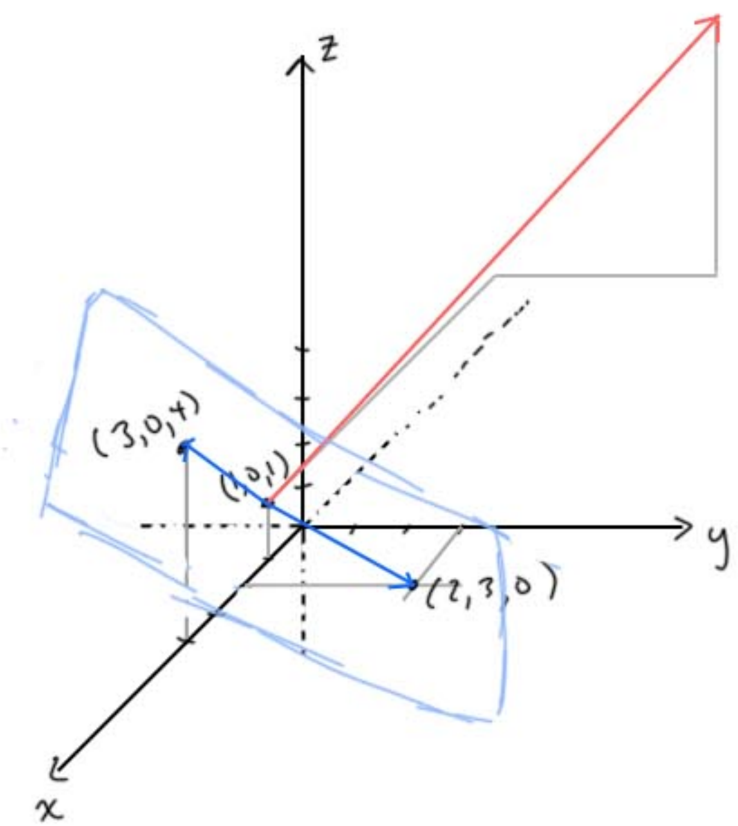
$\overrightarrow{PQ} = \langle -2, 0, -3\rangle \quad \overrightarrow{QR} = \langle 1, 3, -1\rangle$

Use $\overrightarrow{QP} = \langle 2, 0, 3 \rangle$ instead, avoid negatives

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 3 \\ 1 & 3 & -1 \end{vmatrix} =$$

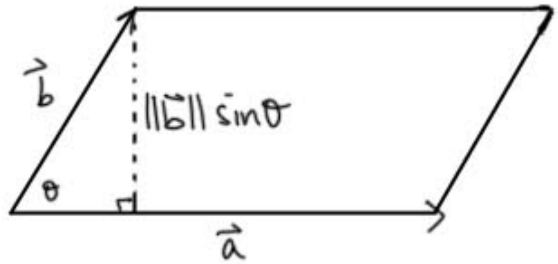
$$= (0-9)\vec{i} - (-2-3)\vec{j} + (6-0)\vec{k} =$$

$$= -9\vec{i} + 5\vec{j} + 6\vec{k}$$ our normal vector to the plane.



With other vectors created from the same points, you may get a vector parallel to $\langle -9, 5, 6 \rangle$

Area of a parallelogram = base · altitude



$= \|\vec{a}\| \, \|\vec{b}\| \sin\theta$

$= \|\vec{a} \times \vec{b}\|$.

Ex. Find the area of the parallelogram formed by $\vec{a} = \langle 2, -1, 5 \rangle$ and $\vec{b} = \langle 0, 2, 3 \rangle$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 5 \\ 0 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 5 \\ 2 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 5 \\ 0 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} \vec{k}$$

$$= (-3 - 10)\vec{i} - (6 - 0)\vec{j} + (4 - 0)\vec{k}$$

$$= -13\vec{i} - 6\vec{j} + 4\vec{k}$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{(-13)^2 + (-6)^2 + 4^2} = \sqrt{169 + 36 + 16}$$

$= \sqrt{221}$ area of parallelogram.