Math 20300
Calculus III

Lesson 3
The Dot Product

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The Dot Product

$\vec{a} = \langle a_1, a_2, a_3 \rangle \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ ← this is a scalar

also in $\mathbb{R}^2$, $\vec{a} = \langle a_1, a_2 \rangle \quad \vec{b} = \langle b_1, b_2 \rangle$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

Ex. $\vec{a} = \langle 3, -1, 0 \rangle \quad \vec{b} = \langle 4, 2, -3 \rangle$

$$\vec{a} \cdot \vec{b} = 3(4) + (-1)(2) + (0)(-3)$$

$$= 12 - 2 + 0 = 10.$$

Properties of Dot Product

1) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2 \qquad \vec{a} = \langle a_1, a_2, a_3 \rangle$

$$\vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2 = \|\vec{a}\|^2 .$$

$$\|\vec{a}\| = \sqrt{a_1^2+a_2^2+a_3^2}$$

2) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ $\quad$ $\vec{a} = \langle a_1, a_2, a_3\rangle$ $\quad$ $\vec{b} = \langle b_1, b_2, b_3\rangle$

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \vec{b}\cdot\vec{a} .$$

3) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ $\quad$ $\vec{c} = \langle c_1, c_2, c_3\rangle$

$$\vec{b}+\vec{c} = \langle b_1+c_1, b_2+c_2, b_3+c_3\rangle$$

$$\vec{a}\cdot(\vec{b}+\vec{c}) = a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$$

$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$

$$= \underbrace{a_1b_1 + a_2b_2 + a_3b_3}_{\vec{a}\cdot\vec{b}} + \underbrace{a_1c_1 + a_2c_2 + a_3c_3}_{\vec{a}\cdot\vec{c}} .$$

4) $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$ $\quad$ c scalar

$c\vec{a} = \langle ca_1, ca_2, ca_3\rangle$ $\quad$ $c\vec{b} = \langle cb_1, cb_2, cb_3\rangle$

$$(c\vec{a})\cdot\vec{b} = ca_1b_1 + ca_2b_2 + ca_3b_3$$

$$= c(a_1b_1 + a_2b_2 + a_3b_3)$$

$$= c(\vec{a}\cdot\vec{b})$$

$$a_1(cb_1) + a_2(cb_2) + a_3(cb_3)$$

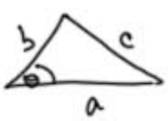
$$= \vec{a}\cdot(c\vec{b})$$

5) $\vec{0}\cdot\vec{a} = 0$ $\qquad\qquad$ $\vec{0} = \langle 0, 0, 0\rangle$

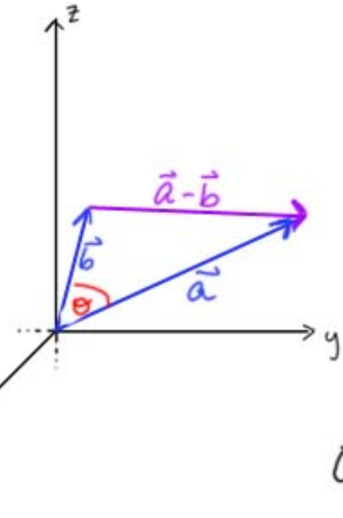
zero vector $\qquad$ zero (scalar)

$$\vec{0}\cdot\vec{a} = 0a_1 + 0a_2 + 0a_3 = 0.$$

Geometric Intepretation of the dot product using Law of Cosines



$c^2 = a^2 + b^2 - 2ab\cos\theta$



$0 \le \theta \le \pi$

$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\cancel{\vec{a}\cdot\vec{a}} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \cancel{\vec{b}\cdot\vec{b}} = \cancel{\vec{a}\cdot\vec{a}} + \cancel{\vec{b}\cdot\vec{b}} - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$-2\,\vec{a}\cdot\vec{b} = -2\,\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta}$$

Ex. $\vec{a} = \langle 2, 0, -2\rangle \quad \vec{b} = \langle 0, -2, 4\rangle$

find The angle between $\vec{a}$ + $\vec{b}$

$$\vec{a}\cdot\vec{b} = 2\cdot 0 + 0(-2) + -2(4) = -8$$

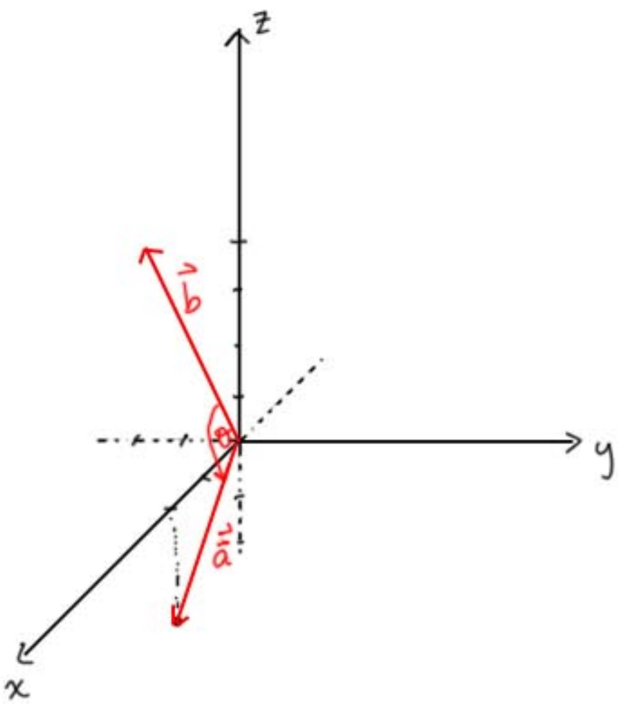
$$\|\vec{a}\| = \sqrt{2^2+0^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\|\vec{b}\| = \sqrt{0^2+(-2)^2+4^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\cos\theta = \frac{-8}{4\sqrt{10}} = \frac{-2}{\sqrt{10}} = \frac{-2\sqrt{10}}{10} = -\frac{\sqrt{10}}{5}$$

$$\theta = \cos^{-1}\left(-\frac{\sqrt{10}}{5}\right) \approx 2.26 \quad \text{or } 129^\circ$$



Orthogonal vectors

nonzero vectors $\vec{a}$ + $\vec{b}$ are said to be <u>orthogonal</u> (perpendicular) if the angle between them is $\frac{\pi}{2}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} \qquad 0 \le \theta \le \pi$$

if $\vec{a}$ & $\vec{b}$ are orthogonal, then

$$\cos\theta = \cos\tfrac{\pi}{2} = 0 = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} \Rightarrow \vec{a}\cdot\vec{b} = 0$$

if $\vec{a}\cdot\vec{b} = 0$ then $\cos\theta = 0$ $\quad 0 \le \theta \le \pi$

$\therefore\ \theta = \frac{\pi}{2}$, $\vec{a}$ & $\vec{b}$ are orthogonal.

$\boxed{\vec{a} \text{ \& } \vec{b} \text{ are orthogonal if and only if } \vec{a}\cdot\vec{b} = 0}$

Ex. Find a vector orthogonal to

$$2\vec{i} - 3\vec{j} + \vec{k} = \langle 2, -3, 1\rangle$$

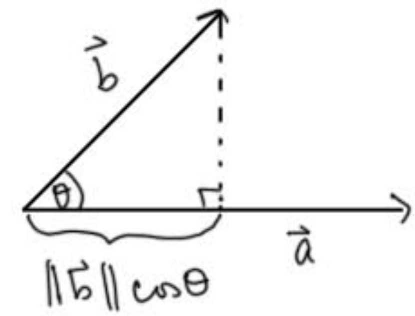
$$\vec{v} = \langle v_1, v_2, v_3\rangle$$

$$2v_1 - 3v_2 + v_3 = 0$$

let $v_1 = 1$
$v_2 = 1$
$v_3 = 1$

$2 - 3 + 1 = 0$ ✓ $\qquad \vec{v} = \langle 1, 1, 1 \rangle$.

Projections



$\|\vec{b}\|\cos\theta$

$= \dfrac{\cancel{\|\vec{b}\|}\ \vec{a}\cdot\vec{b}}{\|\vec{a}\|\ \cancel{\|\vec{b}\|}} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$

call this length the <u>component</u> of $\vec{b}$ along $\vec{a}$

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|a\|}$$

<u>vector projection</u> of $\vec{b}$ onto $\vec{a}$ has length

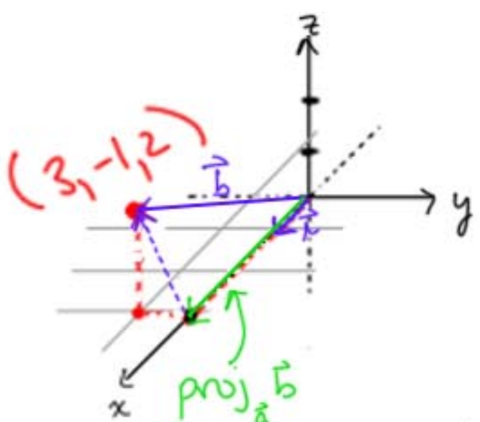
$\text{comp}_{\vec{a}} b$ and is in direction of $\vec{a}$.

so we take unit vector in direction of $\vec{a}$

and scale it by $\text{comp}_{\vec{a}}\vec{b}$

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}\right)\frac{\vec{a}}{\|\vec{a}\|} = \frac{(\vec{a}\cdot\vec{b})\,\vec{a}}{\|\vec{a}\|^2}$$

Ex. Find the distance from point $(3,-1,2)$ to the x axis.



$\vec{b} = \langle 3,-1,2 \rangle$

$\vec{a} = \vec{\imath} = \langle 1,0,0 \rangle$

then $\| \vec{b} - \text{proj}_{\vec{a}} \vec{b} \|$ is the distance we need.

$$\text{proj}_{\vec{a}} \vec{b} = \frac{(\vec{a}\cdot\vec{b})\,\vec{a}}{\|\vec{a}\|^2} = \frac{3\vec{\imath}}{1} = 3\vec{\imath} = \langle 3,0,0 \rangle$$

$$\vec{b} - \text{proj}_{\vec{a}} \vec{b} = \langle 0,-1,2 \rangle$$

$$\| \vec{b} - \text{proj}_{\vec{a}} \vec{b} \| = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}.$$

The orthogonal projection of $\vec{b}$ onto $\vec{a}$:

$$\text{orth}_{\vec{a}} \vec{b} = \vec{b} - \text{proj}_{\vec{a}} \vec{b}.$$