Math 20200
Calculus II

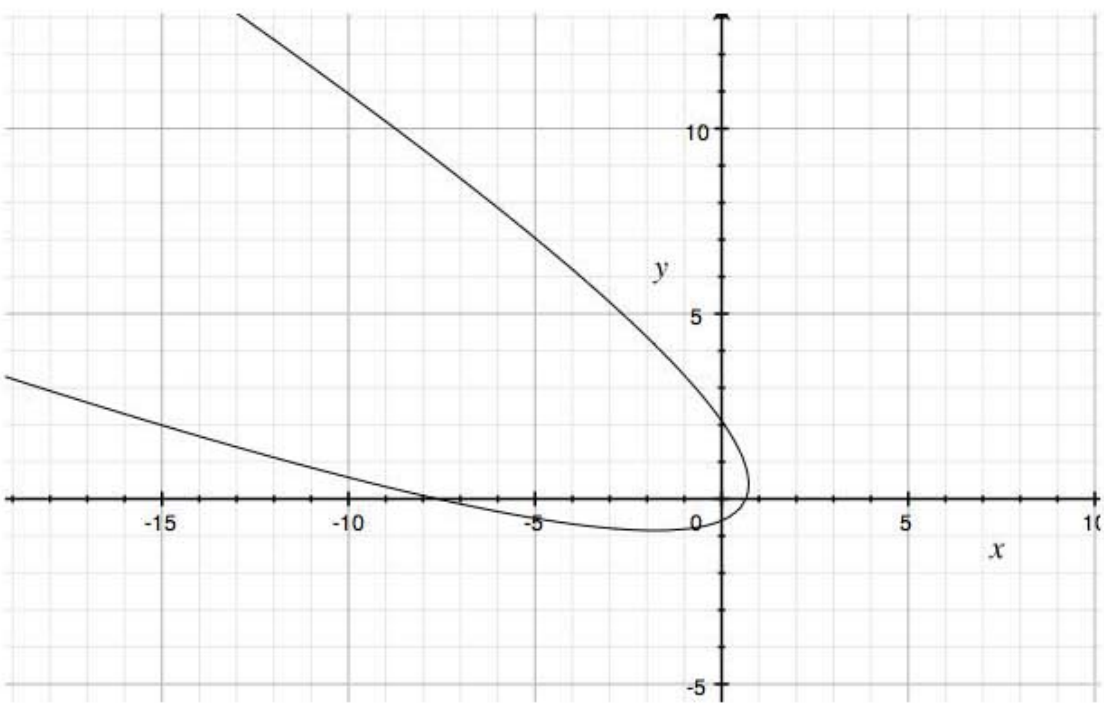
Lesson 28
Rotated Conics

Dr. A. Marchese, The City College of New York

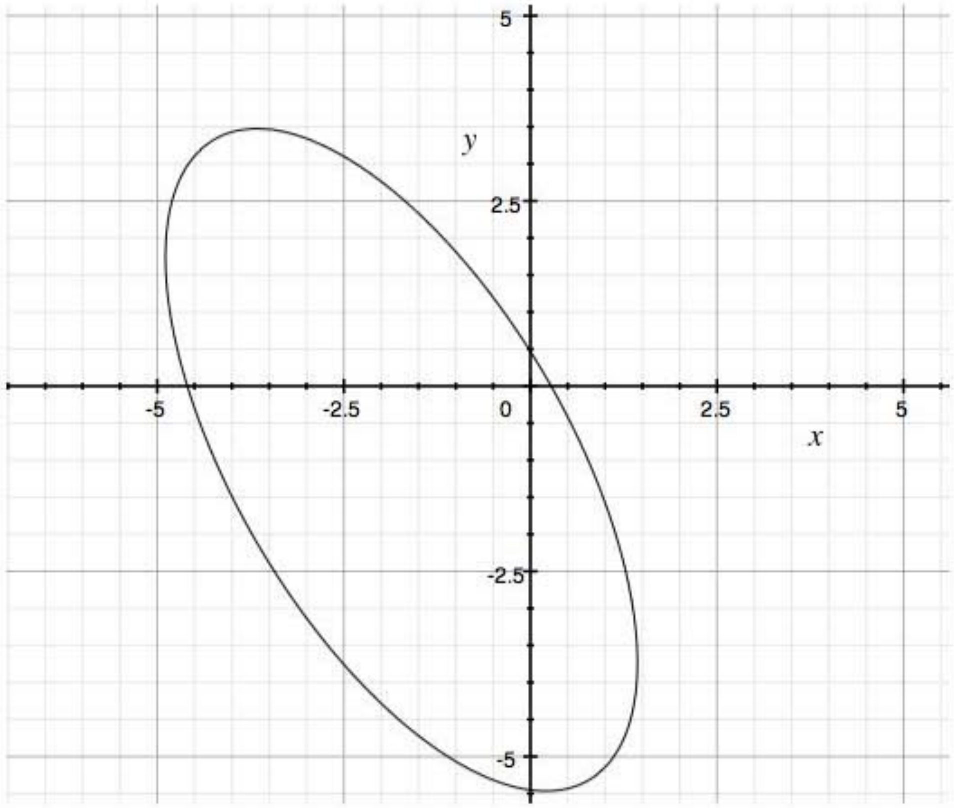
Table of Contents:

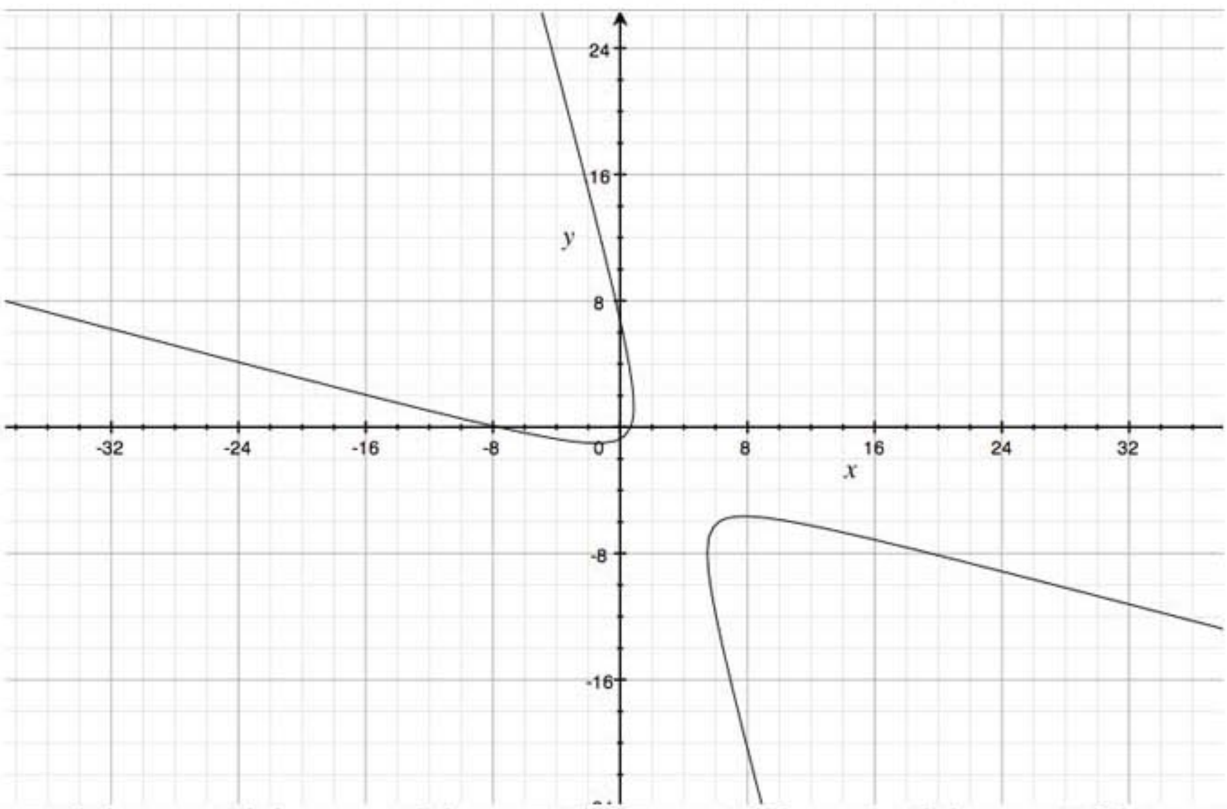
Rotated Conics

In lesson 27, all of the parabolas we discussed had a directrix that was horizontal or vertical. But this curve still fits the definition of a parabola:



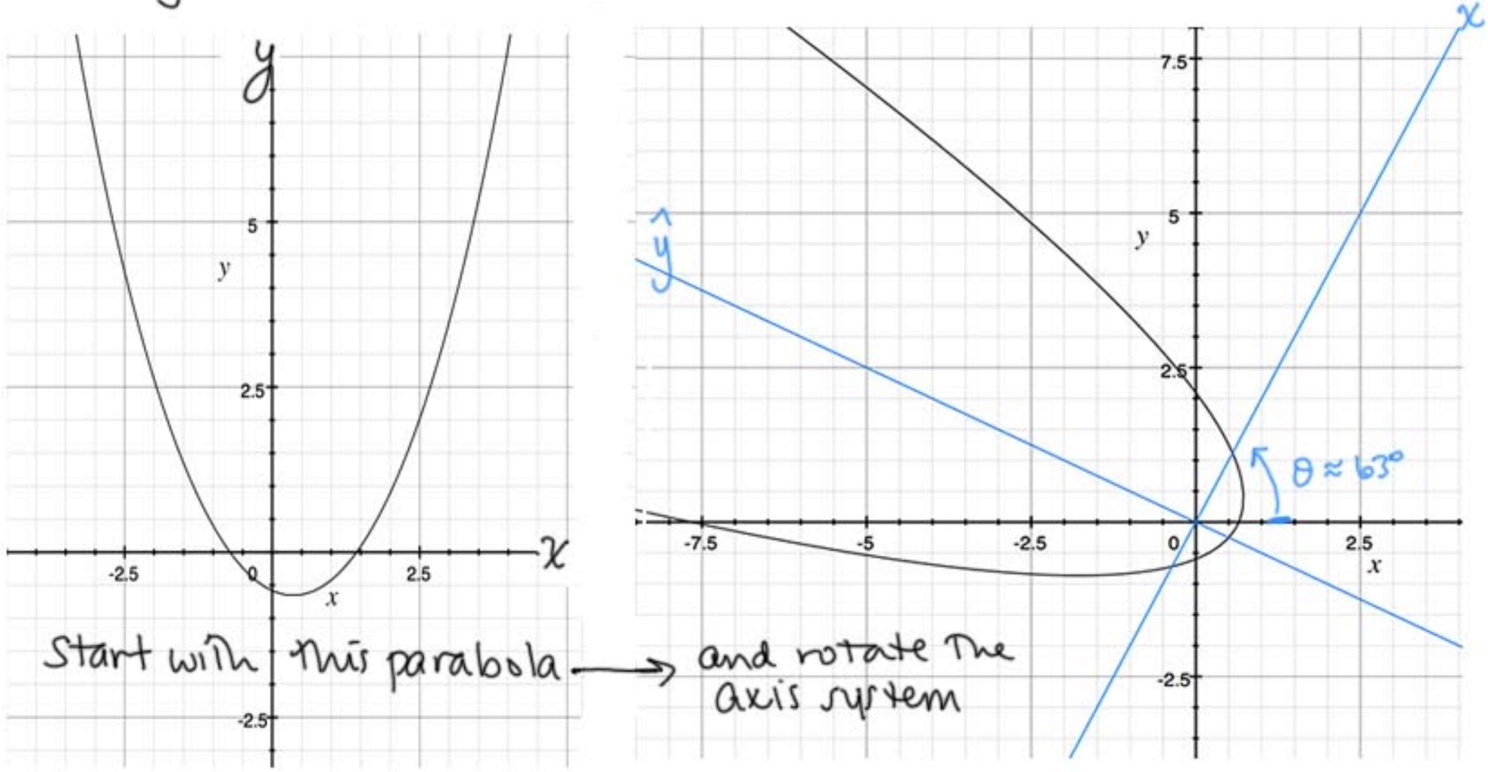
Similarly, the ellipses and hyperbolas we've discussed have all had foci or vertices that lie on a horizontal or vertical line. But the following curves still fit the definition of an ellipse and a hyperbola:





We recognize the above curves as rotations of the conics we have studied so far.

For example, the parabola above can be recognized as a rotation:



Here we rotated the x-y axis system by $\theta \approx 63°$ to form the new $\hat{x}$-$\hat{y}$ axis system.

With regards to the $\hat{x}$-$\hat{y}$ axis system, the new parabola fits all the formulas we learned earlier in the lesson.

$\therefore$ When studying rotated conics, we are interested in 1) how to recognize the equation of a rotated conic section (in x-y form)

2) how to find the angle θ of rotation

3) how to go between x-y and $\hat{x}$-$\hat{y}$ coordinates.

Coordinate Rotation Formulas: (answer to #3)

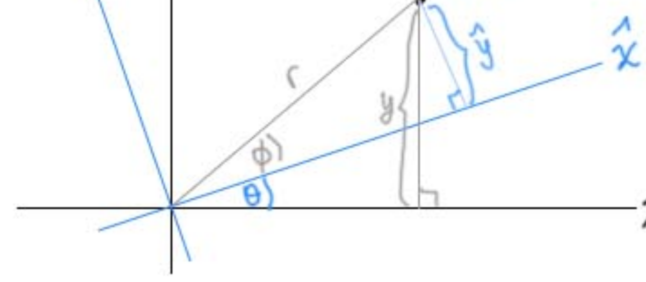
If a rectangular x-y coordinate system is rotated through an angle θ to form the $\hat{x}$-$\hat{y}$ coordinate system, then a point (x,y) will have coordinates $(\hat{x},\hat{y})$ in the new system, where:

$$x = \hat{x}\cos\theta - \hat{y}\sin\theta, \quad y = \hat{x}\sin\theta + \hat{y}\cos\theta$$

and

$$\hat{x} = x\cos\theta + y\sin\theta, \quad \hat{y} = -x\sin\theta + y\cos\theta$$

This comes from:



Now, how to recognize the equation of a rotated conic section (answer to #1):

Recall that earlier, all of the conics had the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

with $AC > 0 \Rightarrow$ ellipse
$AC < 0 \Rightarrow$ hyperbola
$AC = 0 \Rightarrow$ parabola.
(except for degenerate cases)

Rotated conics have the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with $B^2 - 4AC < 0 \Rightarrow$ ellipse
$B^2 - 4AC > 0 \Rightarrow$ hyperbola
$B^2 - 4AC = 0 \Rightarrow$ parabola
(except for degenerate cases)

Note that translating $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ to $\hat{x}$-$\hat{y}$ coordinates gives $\hat{A}\hat{x}^2 + \hat{C}\hat{y}^2 + \hat{D}\hat{x} + \hat{E}\hat{y} + \hat{F} = 0$.
ie, $\hat{B} = 0$

And the angle θ satisfies

(answer to #2 above)

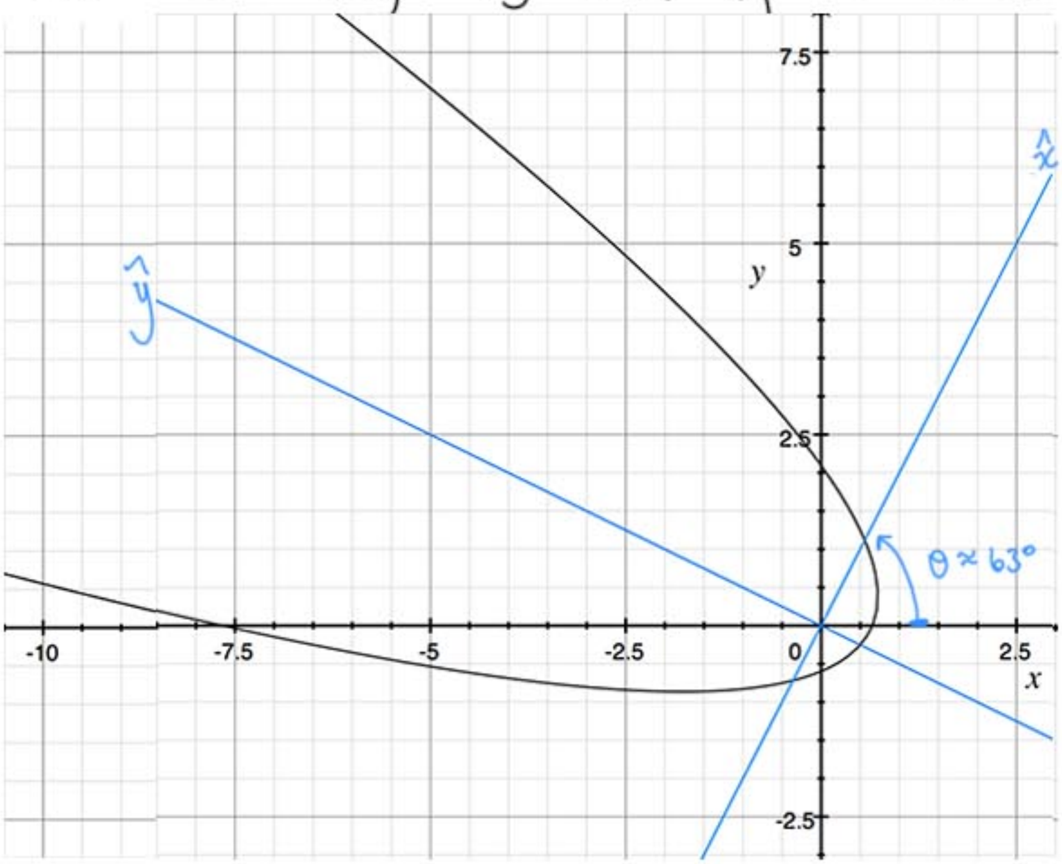
$$\cot 2\theta = \frac{A-C}{B}, \quad 0 < \theta < \frac{\pi}{2}.$$

This comes from translating

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ to $\hat{x}$-$\hat{y}$ coordinates,

and setting the $\hat{B}$ coefficient (of $\hat{x}\hat{y}$) $= 0$.

Ex. $x^2 + 4xy + 4y^2 + 7x - 6y = 5$ is the parabola above



here,

$A = 1$
$B = 4$
$C = 4$

$B^2 - 4AC =$
$16 - 4(1)(4) = 0.$

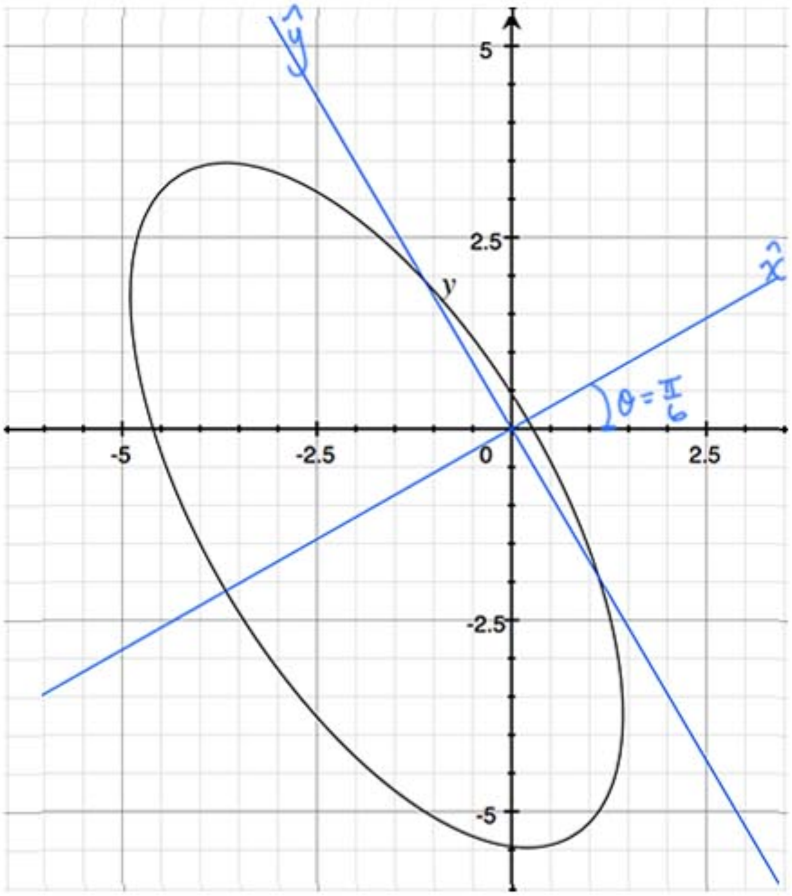
$\cot 2\theta = \frac{1-4}{4} = \frac{-3}{4} \Rightarrow \tan 2\theta = -\frac{4}{3}$

$2\theta = \arctan(-\frac{4}{3}), \quad \theta = \frac{1}{2}\arctan(-\frac{4}{3}) \approx -26.6°$

but we need $0 < \theta < \frac{\pi}{2}$ so add $90°$: $-26.6° + 90° = 63.4°$

Ex. The ellipse above is

$$4x^2 + 2\sqrt{3}\,xy + 2y^2 + 10\sqrt{3}\,x + 10y - 5 = 0$$



$A = 4$

$B = 2\sqrt{3}$

$C = 2$

$B^2 - 4AC =$

$(2\sqrt{3})^2 - 4(4)(2) =$

$12 - 32 < 0$.

and $\cot 2\theta = \dfrac{A - C}{B} = \dfrac{4-2}{2\sqrt{3}} = \dfrac{2}{2\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\tan 2\theta = \sqrt{3}$

$2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$

Ex. Determine the type of conic, and find the angle of rotation: $x^2+4xy+y^2+7x-6y-5=0$.

Work on this problem on your own

$A=1,\ B=4,\ C=1$ $\qquad B^2-4AC = 16-4(1)(1) > 0$

$\Rightarrow$ hyperbola.

$$\cot 2\theta = \frac{A-C}{B} = \frac{1-1}{4} = 0$$

$\cot 2\theta = 0 \Rightarrow \cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$