Math 20200
Calculus II

Lesson 26
Areas and Lengths in Polar Coordinates

Dr. A. Marchese, The City College of New York

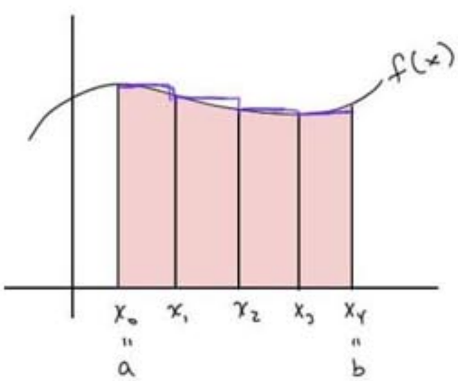
Table of Contents:

pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak

Areas and Lengths in Polar Coordinates

In this lesson we learn to compute the area of regions bounded by polar curves, as well as the arc length of polar curves.

In Cartesian (rectangular) coordinates, we approximate areas using rectangles, and take the number of rectangles to infinity:



$$\text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$$

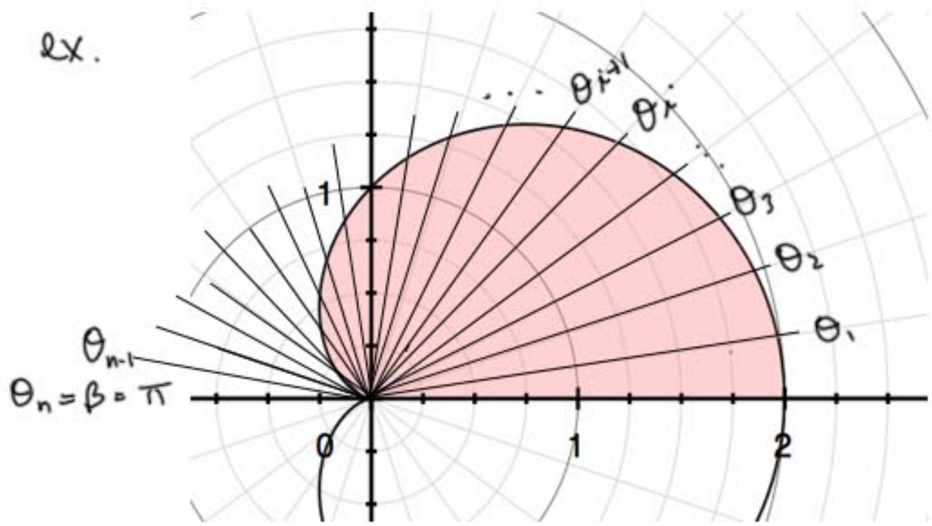
Notice that we we take our x-interval [a,b] and divide it into subintervals.

independent variable

In polar coordinates, θ is the independent variable.

To find the area of a polar region,

ex.



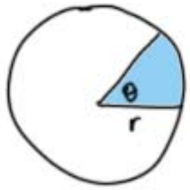
$r = 1 + \cos\theta$

$0 \le \theta \le \pi$

We divide the interval on θ $[\alpha, \beta]$ into subintervals.

$$\text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} \text{area of } i^{th} \text{ sector} \qquad n = \#\text{ sectors}$$

How do we find the area of a circular sector?



entire circle has $\theta = 2\pi$

area $= \pi r^2$

sector of angle θ

let area of sector $= A$

proportion: $\dfrac{\text{angle}}{\text{area}}$ $\quad \dfrac{2\pi}{\pi r^2} = \dfrac{\theta}{A}$

$$\Rightarrow A = \frac{\theta \pi r^2}{2\pi} = \frac{\theta r^2}{2}$$

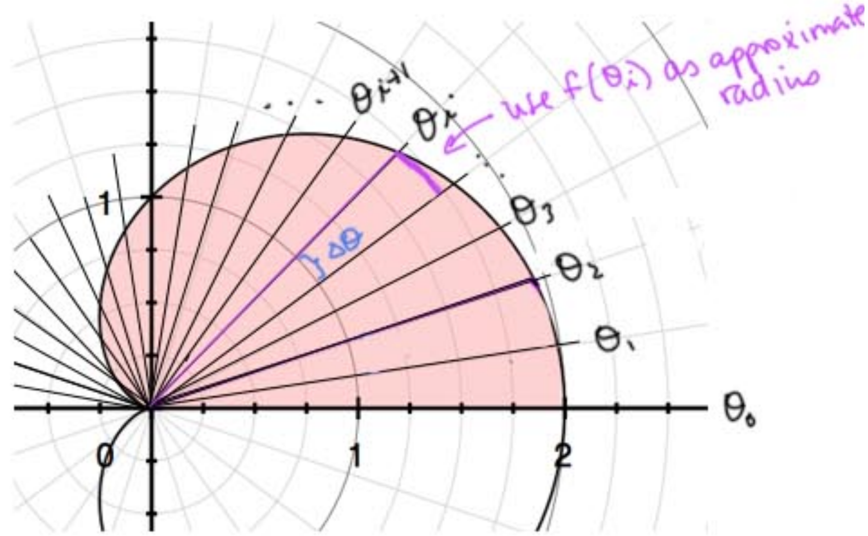
So area of circular sector of angle θ, radius r

$$= \frac{\theta r^2}{2}$$

Then for our area approximation:

angle $= \Delta\theta = \dfrac{\beta - \alpha}{n}$

radius $= f(\theta_i)$
(for i^{th} sector)



$$\therefore \text{ area of } i^{th} \text{ sector} = \Delta\theta \cdot \frac{(f(\theta_i))^2}{2} = \frac{1}{2}(f(\theta_i))^2 \Delta\theta.$$

$$\therefore \quad \text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} \text{area of } i^{th} \text{ sector}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2 \Delta\theta = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta.$$

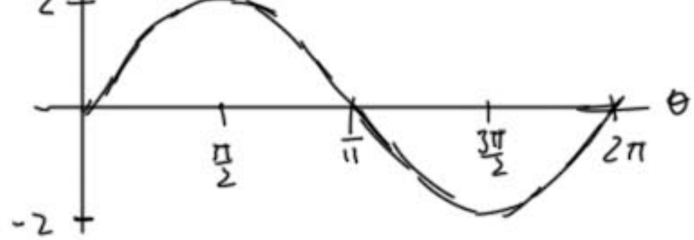
$\therefore$ Area enclosed by $r = f(\theta)$ $\quad \alpha \le \theta \le \beta$

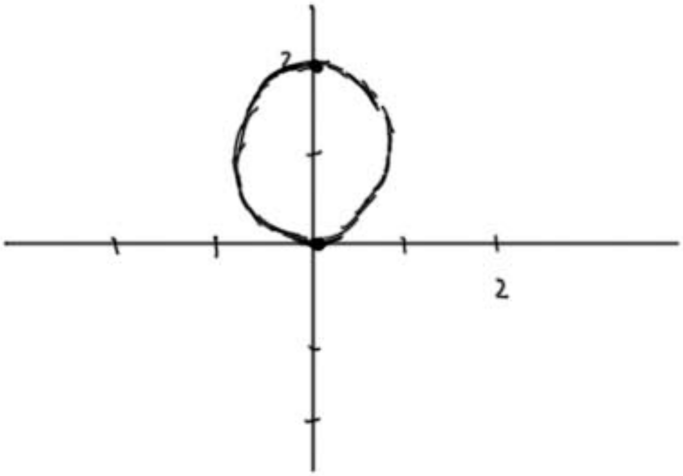
$$= \frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta$$

Ex. Sketch the graph of $r = 2\sin\theta$ and find the area it encloses.

Work on this problem on your own

To sketch: start with $r = 2\sin\theta$ graphed in rectangular form:





$$\underset{\alpha}{\overset{}{0}} \le \theta \le \underset{\beta}{\pi}$$

then area $= \frac{1}{2}\int_0^{\pi} (2\sin\theta)^2 \, d\theta = \frac{1}{2}\cdot 4 \int_0^{\pi} \sin^2\theta \, d\theta =$

$$= 2\int_0^{\pi} \frac{1}{2}(1-\cos 2\theta)\, d\theta = \int_0^{\pi} (1-\cos 2\theta)\, d\theta$$

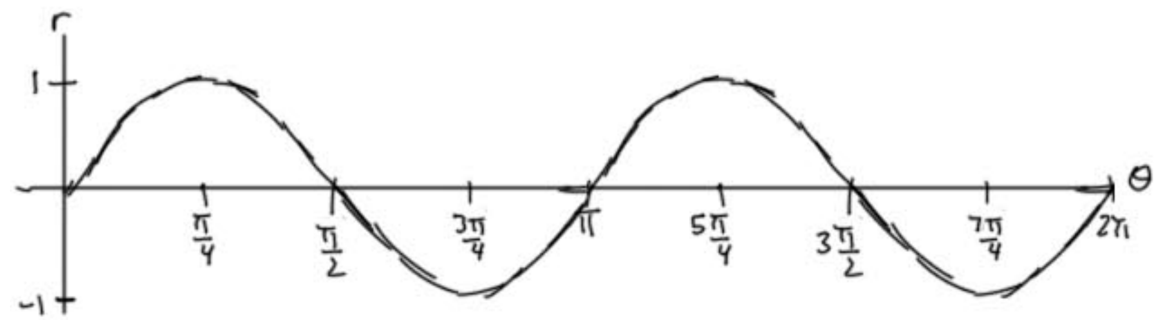
$$= \theta - \frac{1}{2}\sin 2\theta \Big|_0^{\pi} = \left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin(2\cdot 0)\right)$$

$$= \pi .$$

Ex. Sketch the graph of $r = \sin 2\theta$ and find the area it encloses.

Work on this problem on your own

To sketch: start with $r = \sin 2\theta$ graphed in rectangular form:



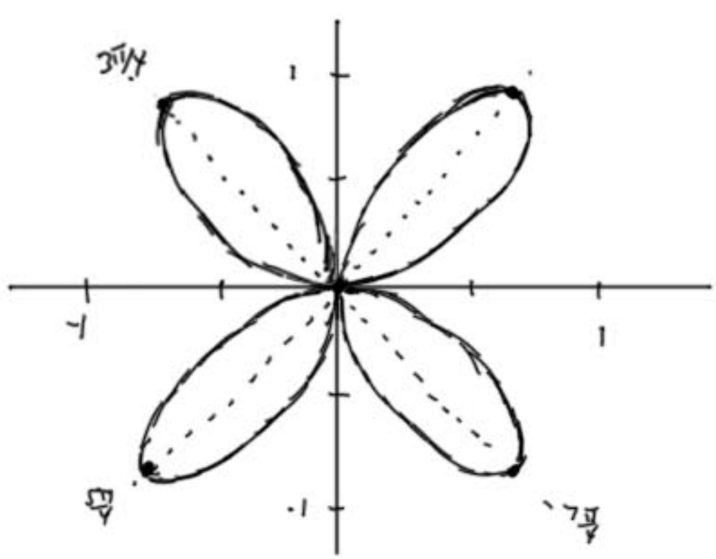
$r = \sin 2\theta$

↑

frequency $= 2$

$pd = \frac{2\pi}{2}$

$= \pi.$



Area enclosed by The graph $= \frac{1}{2}\int_0^{2\pi} (\sin 2\theta)^2 \, d\theta$

OR

Area enclosed by The graph $= 4 \cdot$ (area of one petal)

$$= 4 \cdot \frac{1}{2} \int_0^{\pi/2} (\sin 2\theta)^2 \, d\theta$$

$$= 2 \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4\theta) \, d\theta$$

$$= \int_0^{\pi/2} (1 - \cos 4\theta) \, d\theta$$

$$= \theta - \frac{1}{4}\sin 4\theta \Big|_0^{\pi/2}$$

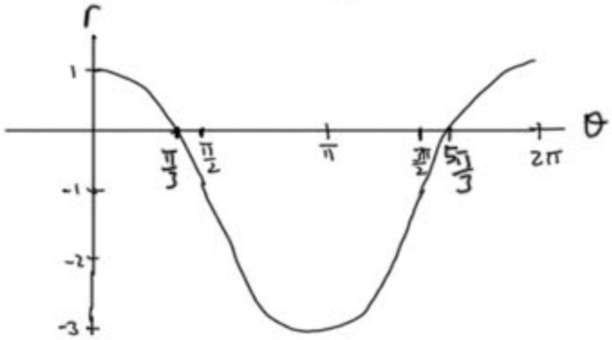
$$= \left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right) - \left(0 - \frac{1}{4}\sin 0\right) = \frac{\pi}{2}.$$

Ex. Sketch the graph of $r = 2\cos\theta - 1$ and find the area of the inner loop.

Work on this problem on your own

To sketch: start with $r = 2\cos\theta - 1$

graphed in rectangular form:

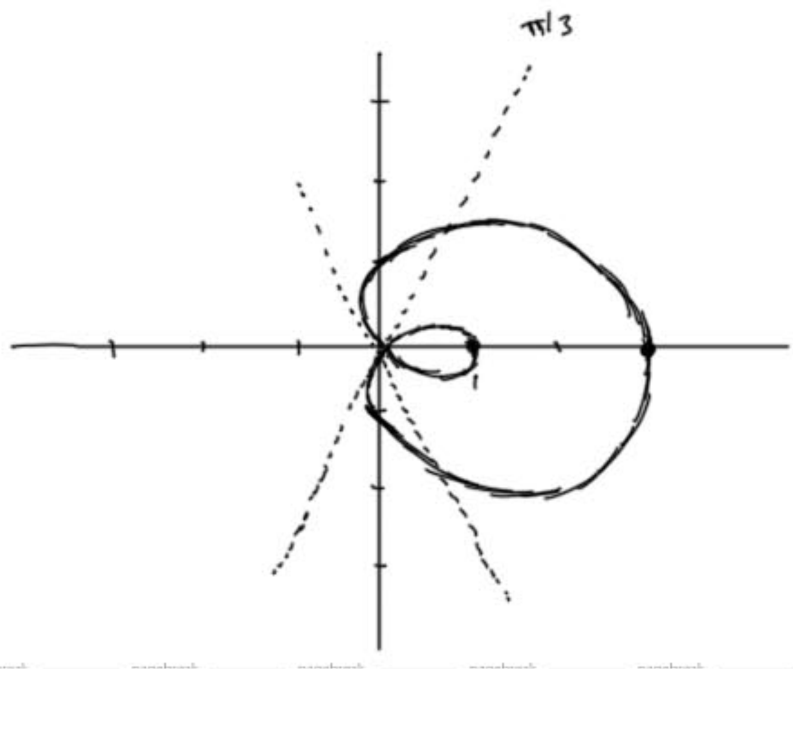


$r = 2\cos\theta - 1 = 0$

$\cos\theta = \frac{1}{2} \quad \theta \in [0, 2\pi]$

$\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

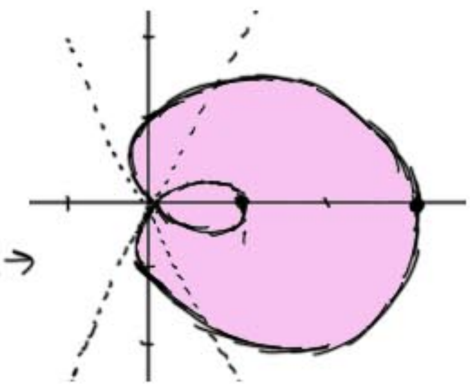
QIV



Be careful with the bounds on the integral:

$$\frac{1}{2}\int_{5\pi/3}^{\pi/3}(2\cos\theta-1)^2\,d\theta = -\frac{1}{2}\int_{\pi/3}^{5\pi/3}(2\cos\theta-1)^2\,d\theta$$

=



negative area →

NOT correct interval.

Either $\frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos\theta-1)^2\,d\theta$ OR $2\cdot\frac{1}{2}\int_{0}^{\pi/3}(2\cos\theta-1)^2\,d\theta$

by symmetry of the inner loop.

$$2\cdot\frac{1}{2}\int_{0}^{\pi/3}(2\cos\theta-1)^2\,d\theta = \int_{0}^{\pi/3}\big(\underbrace{4\cos^2\theta}_{4\cdot\frac{1}{2}(1+\cos 2\theta)}-4\cos\theta+1\big)\,d\theta$$

$$= \int_{0}^{\pi/3}\big(2(1+\cos 2\theta)-4\cos\theta+1\big)\,d\theta$$

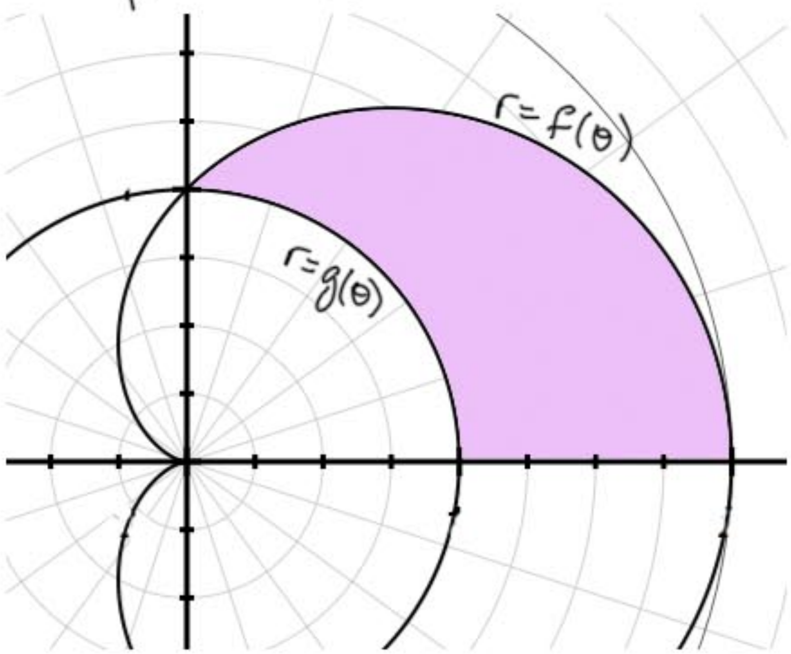
$$= \int_{0}^{\pi/3}(2\cos 2\theta-4\cos\theta+3)\,d\theta =$$

$$= \sin 2\theta - 4\sin\theta + 3\theta \Big|_0^{\pi/3}$$

$$= \left(\sin\tfrac{2\pi}{3} - 4\sin\tfrac{\pi}{3} + 3\cdot\tfrac{\pi}{3}\right) - \left(\sin(0) - 4\sin(0) + 3(0)\right)$$

$$= \frac{\sqrt{3}}{2} - 4\cdot\frac{\sqrt{3}}{2} + \pi = \frac{-3\sqrt{3}}{2} + \pi.$$

Area between polar curves :



The area between $r = f(\theta)$ and $r = g(\theta)$ $\left(f(\theta) \geq g(\theta)\right)$

over $\alpha \leq \theta \leq \beta$

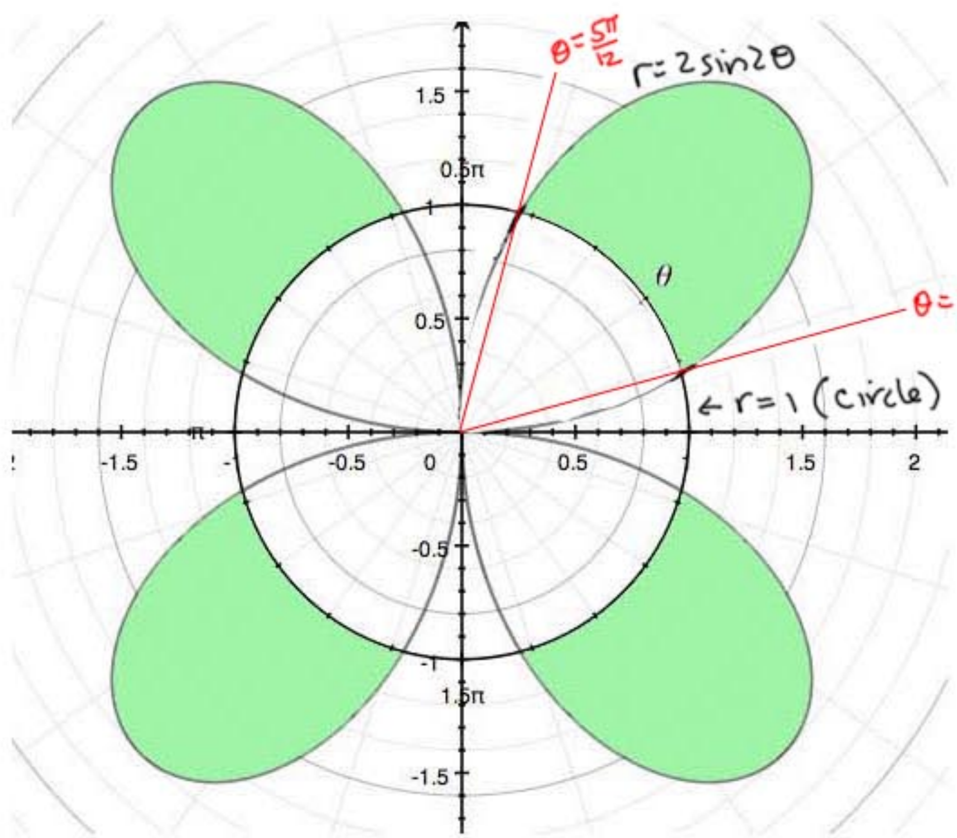
$$= \frac{1}{2}\int_\alpha^\beta \left(f(\theta)\right)^2 d\theta - \frac{1}{2}\int_\alpha^\beta \left(g(\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta}\left((f(\theta))^2 - (g(\theta))^2\right)\, d\theta.$$

Ex. Find the area inside $r = 2\sin 2\theta$ and outside $r = 1$.

Work on this problem on your own

(graph and shade region)



we need to find the θ-values for which the curves intersect.

Set $2\sin 2\theta = 1$

solve for θ.

$$2\sin 2\theta = 1$$

$$\sin 2\theta = \tfrac{1}{2}$$

$$2\theta = \frac{\pi}{6} + 2\pi k \quad k \in \mathbb{Z} \qquad \Rightarrow \qquad \theta = \frac{\pi}{12} + \pi k$$

$$\text{or} \quad 2\theta = \frac{5\pi}{6} + 2\pi k \quad k \in \mathbb{Z} \qquad \qquad \theta = \frac{5\pi}{12} + \pi k$$

$$\text{area} = 4 \cdot \frac{1}{2} \int_{\pi/12}^{5/12} \left((2\sin 2\theta)^2 - 1^2 \right) d\theta$$

$$\sin^2 2\theta = \tfrac{1}{2}(1 - \cos 4\theta)$$

$$4\sin^2 2\theta = \underbrace{4 \cdot \tfrac{1}{2}}_{2}(1 - \cos 4\theta)$$

$$= 2 - 2\cos 4\theta$$

$$= 2 \int_{\pi/12}^{5\pi/12} \left(4\sin^2 2\theta - 1 \right) d\theta$$

$$= 2 \int_{\pi/12}^{5\pi/12} \left(2 - 2\cos 4\theta - 1 \right) d\theta$$

$$= 2 \int_{\pi/12}^{5\pi/12} (1 - 2\cos 4\theta)\, d\theta = 2\left(\theta - \frac{2}{4}\sin 4\theta \right) \Big|_{\pi/12}^{5\pi/12}$$

$$= 2\left[\left(\frac{5\pi}{12} - \frac{1}{2}\sin\frac{5\pi}{3} \right) - \left(\frac{\pi}{12} - \frac{1}{2}\sin\frac{\pi}{3} \right) \right]$$

$$= 2\left[\left(\frac{5\pi}{12} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) \right) - \left(\frac{\pi}{12} - \frac{1}{2}\frac{\sqrt{3}}{2} \right) \right]$$

$$= 2\left[\frac{5\pi}{12} + \frac{\sqrt{3}}{4} - \frac{\pi}{12} + \frac{\sqrt{3}}{4}\right]$$

$$= 2\left(\frac{\cancel{4}\pi}{\cancel{12}\,3} + \frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3}.$$

Arc Length of polar curves:

As with slopes, we regard the polar curve

$r = f(\theta)$ as a parametric curve: $\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$.

then the length of $r = f(\theta)$ $\alpha \le \theta \le \beta$ is

$$S = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \qquad (\text{Lesson } 23)$$

and $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 +$

$$\left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2$$

$$= (f'(\theta))^2 \cos^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta + (f(\theta))^2 \sin^2\theta +$$

$$+ (f'(\theta))^2 \sin^2\theta + 2f(\theta)f'(\theta)\sin\theta\cos\theta + (f(\theta))^2\cos^2\theta$$

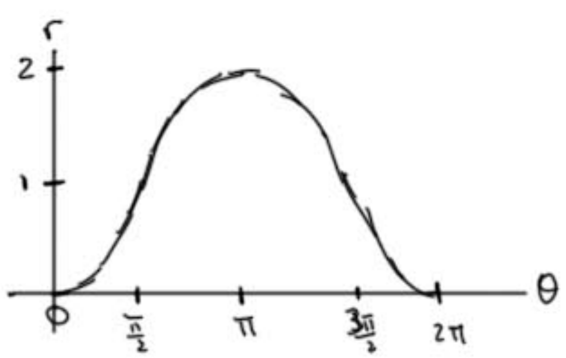
$$= (f'(\theta))^2 + (f(\theta))^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

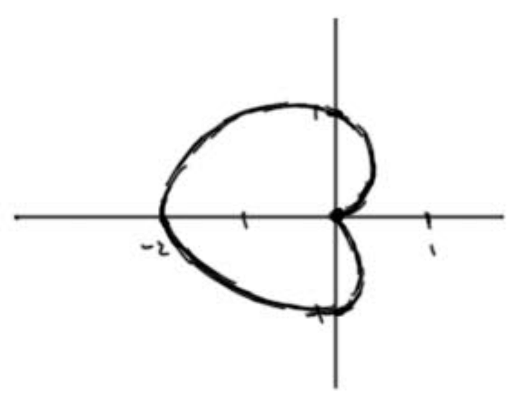
$\therefore$ the length of $r = f(\theta)$ $\alpha \le \theta \le \beta$ is

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Ex. Find the length of the cardioid $r = 1 - \cos\theta$.

Notice we aren't given bounds on θ, so we have to examine the graph of $r = 1-\cos\theta$ and find bounds on θ.





$r = 1-\cos\theta$

$\frac{dr}{d\theta} = \sin\theta$

$$S = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\ d\theta$$

$$= \int_0^{2\pi} \sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\ d\theta$$

$$= \int_0^{2\pi} \sqrt{2-2\cos\theta}\ d\theta$$

we know

$\frac{1}{2}(1-\cos 2x) = \sin^2 x$

$2(1-\cos 2x) = 4\sin^2 x$

$2(1-\cos\theta) = 4\sin^2\left(\frac{\theta}{2}\right)$

$$= \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\ d\theta$$ and $\sin\frac{\theta}{2} \geq 0$ on $[0, 2\pi]$

$$= \int_0^{2\pi} 2\sin\left(\frac{\theta}{2}\right) d\theta = -4\cos\left(\frac{\theta}{2}\right)\Big|_0^{2\pi}$$

$$= (-4\cos\pi) - (-4\cos 0)$$

$$= -4(-1) - (-4) = 8.$$