Math 20200
Calculus II

Lesson 25
Polar Curves

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Polar Curves

In rectangular (Cartesian) coordinates,

$x = c$ (constant) is a vertical line

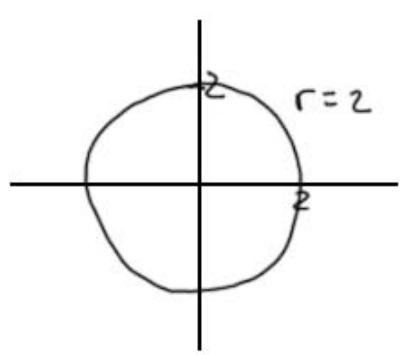
$y = d$ (constant) is a horizontal line.

In polar coordinates,

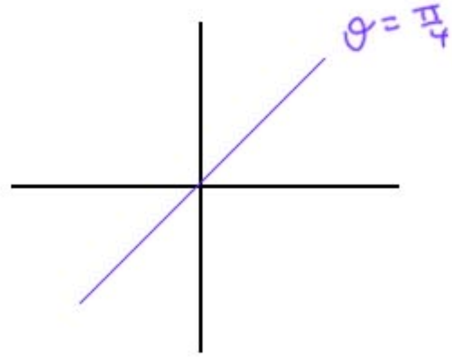
$r = c$ (constant) is a circle of radius r centered at the origin

$\theta = d$ (constant) is a line through the origin at angle θ.

Ex. $r = 2$ is the set of points in the plane whose distance to the origin is 2.

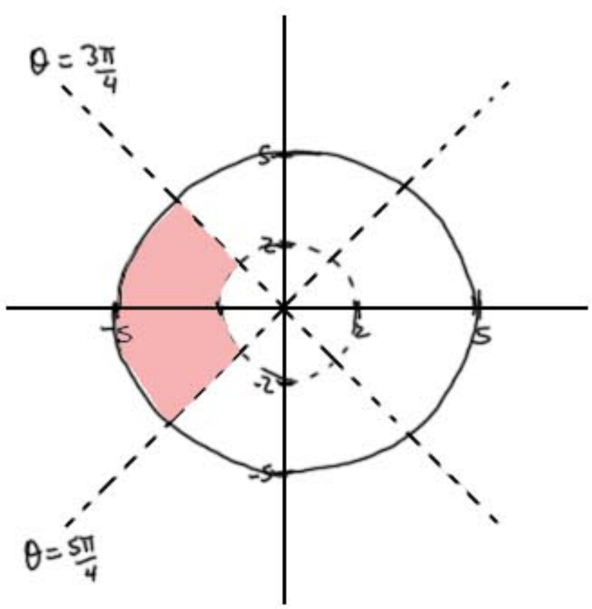


Ex. $\theta = \frac{\pi}{4}$ is the set of points that can be described as $(r, \frac{\pi}{4})$ (with positive or negative r-values).



Ex. Sketch the $\underbrace{\text{region}}_{\substack{\text{2D} \\ \text{shade in}}}$ $2 < r \leq 5$ $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$

Work on this problem on your own



$2 < r \leq 5$ $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$

Ex. Find a Cartesian equation for the curve:

$$r = 2\sin\theta + 2\cos\theta$$

we know $x = r\cos\theta$ & $y = r\sin\theta$

so multiply by r: $\quad r^2 = 2r\sin\theta + 2r\cos\theta$

$$x^2 + y^2 = 2y + 2x$$

then $\quad x^2 - \underbrace{2x}_{\div 2} + 1 + y^2 - 2y + 1 = 0 + 1 + 1$

complete the squares

$$(x-1)^2 + (y-1)^2 = 2$$

Circle centered at $(1,1)$ with radius $= \sqrt{2}$.

Ex. Find a Cartesian equation for the curve:

$$r = \tan\theta \sec\theta$$

Work on this problem
on your own

$$r = \frac{y}{x} \cdot \frac{1}{\cos\theta}$$ multiply both sides by $\cos\theta$

$$r\cos\theta = \frac{y}{x}$$

$$x = \frac{y}{x} \Rightarrow y = x^2.$$

Ex. Find a polar equation for the curve:

$$x + y = 9$$

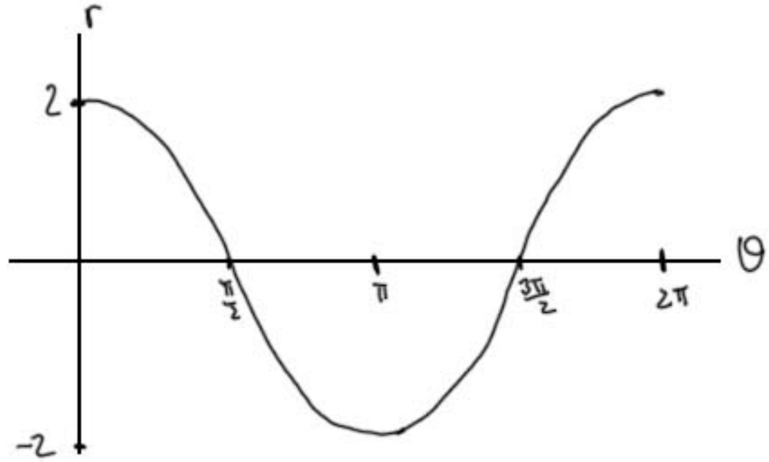
$r\cos\theta + r\sin\theta = 9$ want $r = f(\theta)$ if possible

$$r(\cos\theta + \sin\theta) = 9 \Rightarrow r = \frac{9}{\cos\theta + \sin\theta}.$$

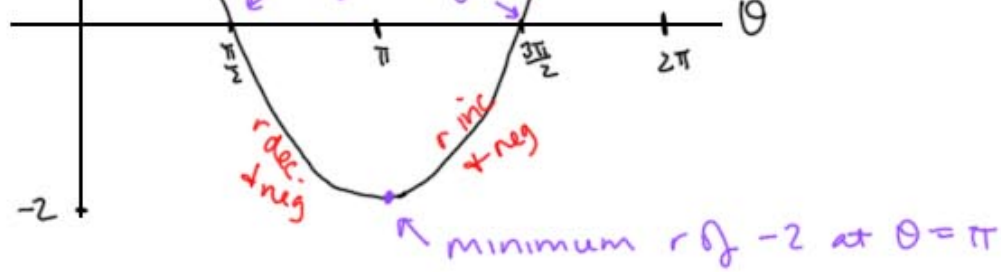
Plotting Polar Curves

Ex. Sketch the graph of $r = 2\cos\theta$.

step① graph the function in rectangular (r, θ) coordinates:

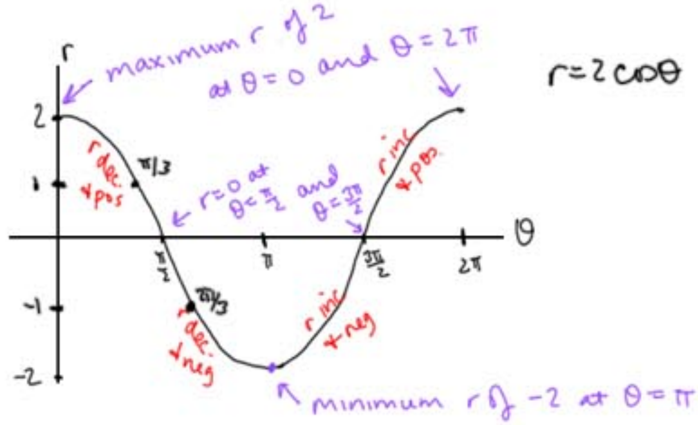


step② identify the important features of the graph, ie find the θ-values for which r = 0, and find the θ-values where r is maximized and minimized, note the intervals on θ where r is positive and negative, increasing and decreasing.

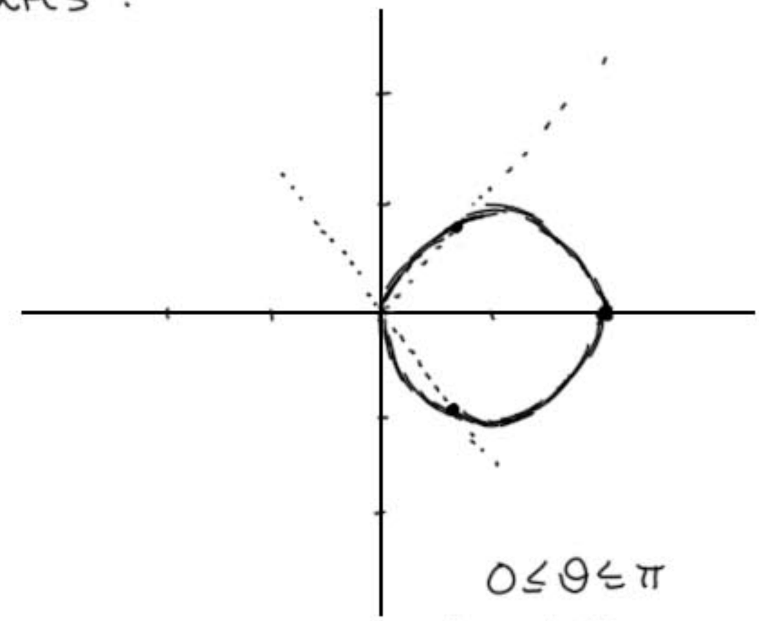


step③ transfer this information to a graph

in polar coordinates :



$r = 2\cos\theta$



$0 \le \theta \le \pi$
complete graph

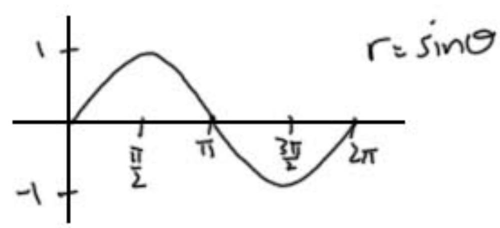
Ex. Plot the polar curve : $r = 1 - \sin\theta$

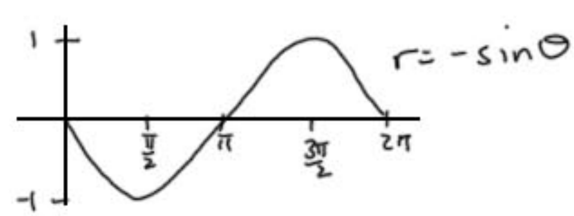
Work on this problem on your own

first graph $r = \sin\theta$

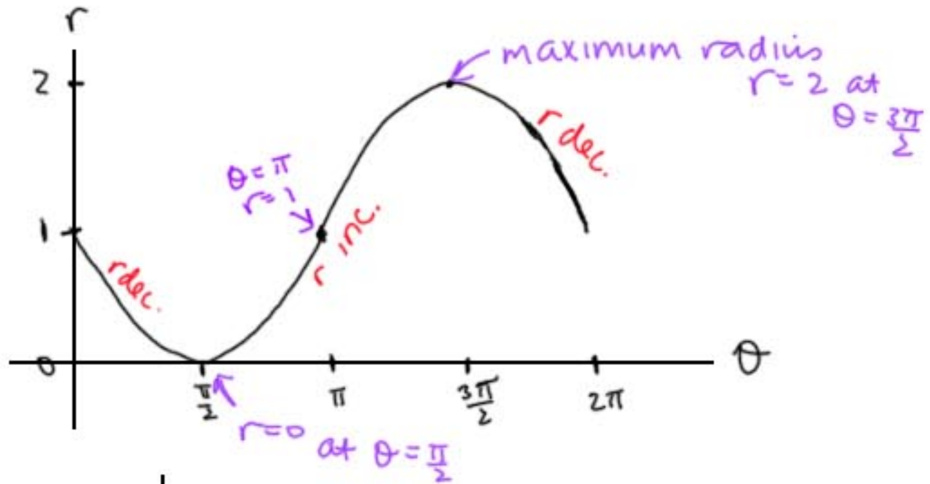
then $r = -\sin\theta$

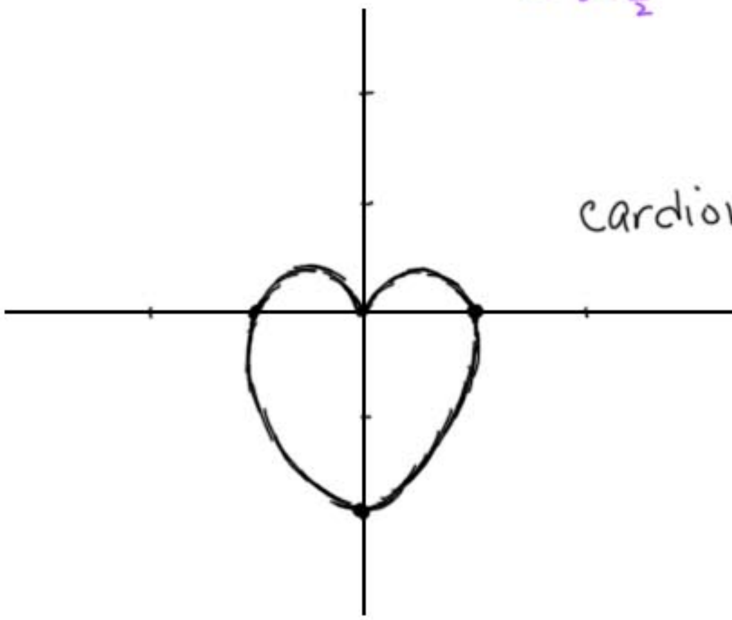
then $r = -\sin\theta + 1$





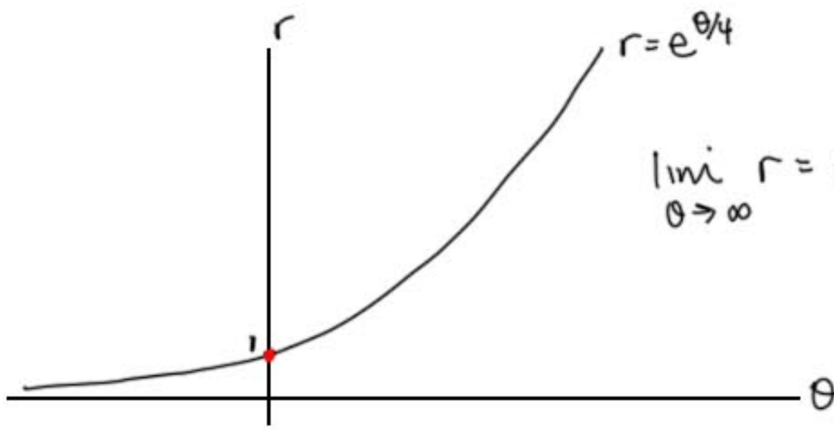
$r = 1 - \sin\theta$





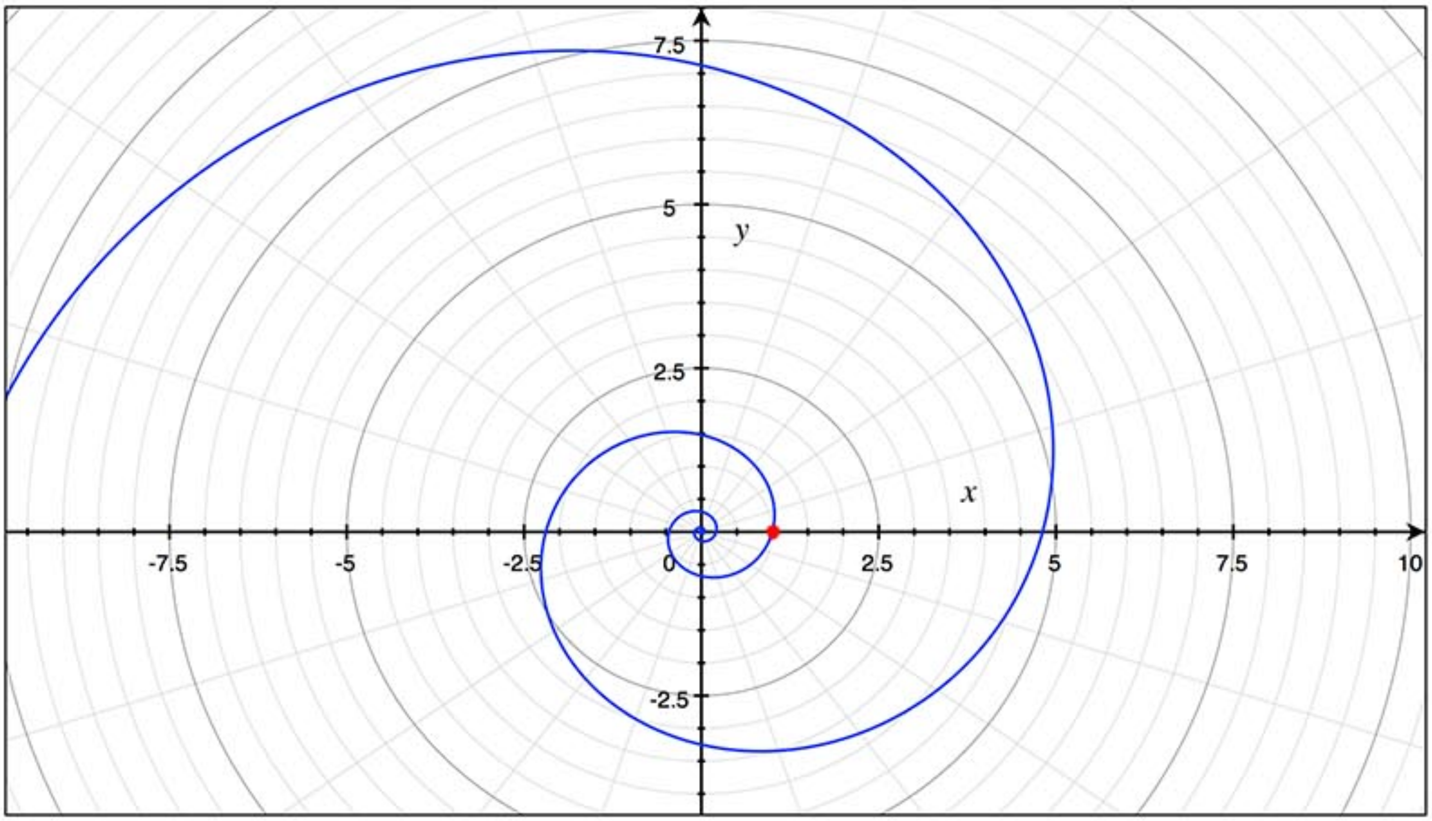
cardioid

Ex. $r = e^{\theta/4}$



$\lim_{\theta \to \infty} r = \infty$

$\lim_{\theta \to -\infty} r = 0$



More graphing in lesson 26.

Finding the slope of a polar curve:

Remember that for $r = f(\theta)$,

$$f'(\theta) = \frac{dr}{d\theta} = \text{change in } r \text{ with respect to a change in } \theta.$$

does not give the slope of the curve plotted in polar coordinates.

The slope is still $\frac{dy}{dx}$ = change in y with respect to a change in x.

So we think of the polar curve as a parametric curve, and find $\frac{dy}{dx}$ as in Lesson 23.

So for $r = f(\theta)$, we know

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Ex. Find the slope of the tangent line to $r = 1-\sin\theta$ at $\theta = \pi$.

$$f(\theta) = 1 - \sin\theta$$

$$f'(\theta) = -\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{-\cos\theta\sin\theta + (1-\sin\theta)\cos\theta}{-\cos^2\theta - (1-\sin\theta)\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{-\cos\pi\sin\pi + (1-\sin\pi)\cos\pi}{-\cos^2\pi - (1-\sin\pi)\sin\pi}$$

$$= \frac{-(-1)(0) + (1-0)(-1)}{-(-1)^2 - (1-0)(0)} = \frac{-1}{-1} = 1$$

we saw This graph above: