Math 20200
Calculus II

Lesson 23
Calculus with Parametric Curves

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Calculus with Parametric Curves

Suppose we have a curve defined parametrically with $x = f(t)$, $y = g(t)$, f and g differentiable in t, and y a differentiable function of x.

chain rule: for $y = y(x(t))$

$$\frac{d}{dt}(y) = y'(x(t)) \cdot x'(t)$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{solve for } \frac{dy}{dx}$$

Slope of the curve in the x-y plane $\quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ at any t-value, provided $\frac{dx}{dt} \neq 0$

and the second derivative: $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\underbrace{\dfrac{dy}{dx}}_{y}\right)$

plug $\frac{dy}{dx}$ in for y $\quad \left(\dfrac{d\boxed{y}}{dx} = \dfrac{\frac{d\boxed{y}}{dt}}{\frac{dx}{dt}} \right)$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Ex. $\begin{cases} x = \sqrt{t} \\ y = e^t \end{cases}$ $t \geq 0$

a) find $\dfrac{dy}{dx}$

$\left.\dfrac{dy}{dx}\right|_{t=0}$

$\left.\dfrac{dy}{dx}\right|_{t=1}$

b) find $\dfrac{d^2y}{dx^2}$ and discuss concavity of curve.

a) $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{e^t}{\frac{1}{2\sqrt{t}}} = 2\sqrt{t}e^t$

$\dfrac{dy}{dt} = e^t$

$\left.\dfrac{dy}{dx}\right|_{t=0} = 2\sqrt{0}\,e^0 = 0$ = slope of curve at $t=0$

$\left.\dfrac{dy}{dx}\right|_{t=1} = 2\sqrt{1}\,e^1 = 2e$ = slope of curve at $t=1$

b) $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{\frac{d}{dt}\left(2\sqrt{t}\,e^t\right)}{\frac{1}{2\sqrt{t}}} = \dfrac{\overbrace{\frac{1}{\sqrt{t}}e^t + 2\sqrt{t}\,e^t}^{\text{product rule}}}{\frac{1}{2\sqrt{t}}}$

$= 2\sqrt{t}\left(\frac{1}{\sqrt{t}}e^t + 2\sqrt{t}\,e^t\right) = 2e^t + 4te^t$

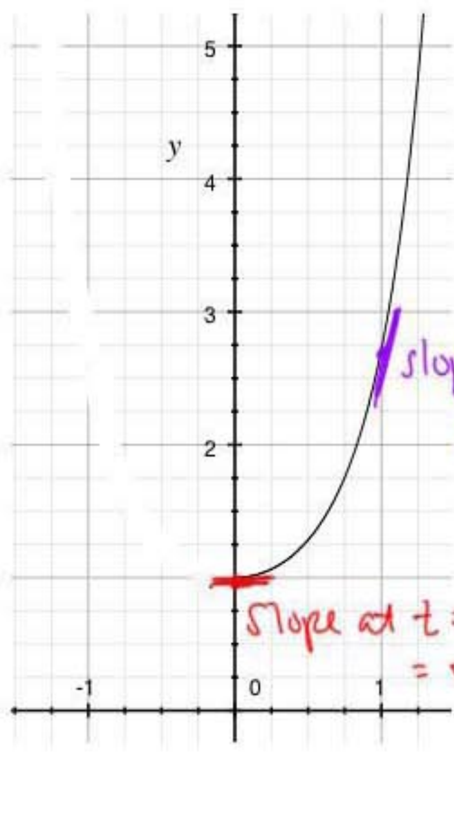
$$= \underbrace{e^t}_{>0}\underbrace{\underbrace{(2+4t)}_{\geq 2}}_{>0} > 0 \quad \therefore \text{ curve is concave up } \forall t \geq 0$$

to graph, let's eliminate the parameter.

$$x = \sqrt{t} \Rightarrow t = x^2 \qquad t \geq 0$$

$$y = e^t = e^{x^2} \qquad x = \sqrt{t} \geq 0$$

$$y = e^{x^2} \text{ for } x \geq 0$$



Concave up for $t \geq 0$

slope at $t=1$ is $2e$

Slope at $t=0$ $= 0$

$t=1$

$x = \sqrt{1} = 1$

Ex. $\begin{cases} x = t^2 - 1 \\ y = \sin 2t \end{cases}$

a) sketch curve for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

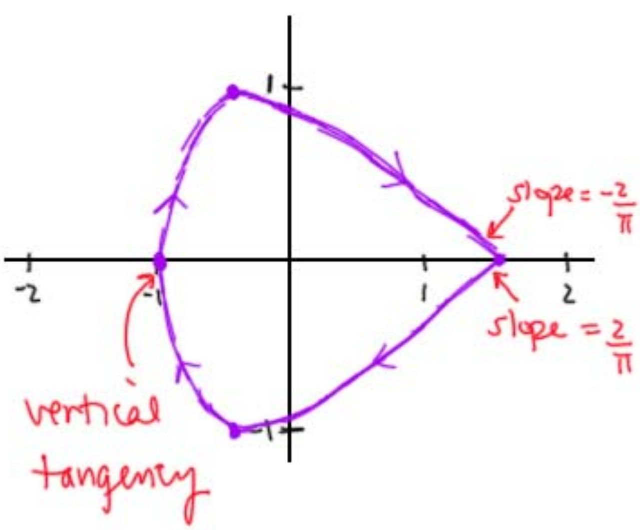
b) find $\frac{dy}{dx}$ for $t = -\frac{\pi}{2}$

$t = \frac{\pi}{2}$

$t = 0$

$x = t^2 - 1$

$y = \sin 2t$



t	$x = t^2 - 1$	$y = \sin 2t$
$-\frac{\pi}{2}$	$\frac{\pi^2}{4} - 1 \approx \frac{6}{4} = \frac{3}{2}$	$\sin(-\pi) = 0$
$-\frac{\pi}{4}$	$\frac{\pi^2}{16} - 1 \approx \frac{-6}{16} = -\frac{3}{8}$	$\sin(-\frac{\pi}{2}) = -1$
0	-1	0
$\frac{\pi}{4}$	$\frac{\pi^2}{16} - 1 \approx -\frac{3}{8}$	1
$\frac{\pi}{2}$	$\frac{\pi^2}{4} - 1 \approx \frac{3}{2}$	0

b) find $\frac{dy}{dx}$ at $t = \frac{-\pi}{2}$
$t = \frac{\pi}{2}$
$t = 0$

$\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = 2\cos 2t$

$$\frac{dy}{dx} = \frac{2\cos 2t}{2t} = \frac{\cos 2t}{t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{-\pi}{2}} = \frac{-1}{\frac{-\pi}{2}} = \frac{2}{\pi}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{-1}{\pi/2} = -\frac{2}{\pi}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{1}{0}$$ infinite slope

DNE vertical tangency

Ex. Find the points on the curve where the tangent line is horizontal or vertical:

$$\begin{cases} x = \sin t \\ y = \cos(3t) \end{cases} \quad \text{for } -\pi \le t \le \pi .$$

horizontal tangent line: $\frac{dy}{dx} = 0$

vertical tangent line: $\frac{dy}{dx} = \frac{\text{nonzero}}{\text{zero}} \quad (\pm\infty)$

$\frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -3\sin(3t) \qquad \frac{dy}{dx} = \frac{-3\sin(3t)}{\cos t}$

horizontal: $\frac{-3\sin(3t)}{\cos t} = 0 \Rightarrow -3\sin(3t) = 0$

$\sin(3t) = 0$

$3t = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

(notice $\cos t \neq 0$ at any $\Rightarrow$) $t = 0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\pi$.

we are asked for the points, not the t-values:

$t=0$: $x=0$, $y=1$ $\quad (0,1)$

$t=\frac{\pi}{3}$: $x=\frac{\sqrt{3}}{2}$, $y=-1$ $\quad \left(\frac{\sqrt{3}}{2}, -1\right)$

$t=-\frac{\pi}{3}$: $x=-\frac{\sqrt{3}}{2}$, $y=-1$ $\quad \left(-\frac{\sqrt{3}}{2}, -1\right)$

$t=\frac{2\pi}{3}$: $x=\frac{\sqrt{3}}{2}$, $y=1$ $\quad \left(\frac{\sqrt{3}}{2}, 1\right)$

$t=-\frac{2\pi}{3}$: $x=-\frac{\sqrt{3}}{2}$, $y=1$ $\quad \left(-\frac{\sqrt{3}}{2}, 1\right)$

$t=\pi$:

$x=0$, $y=-1$

$(0,-1)$

$t=-\pi$:

$x=0$, $y=-1$

$(0,-1)$

(same)

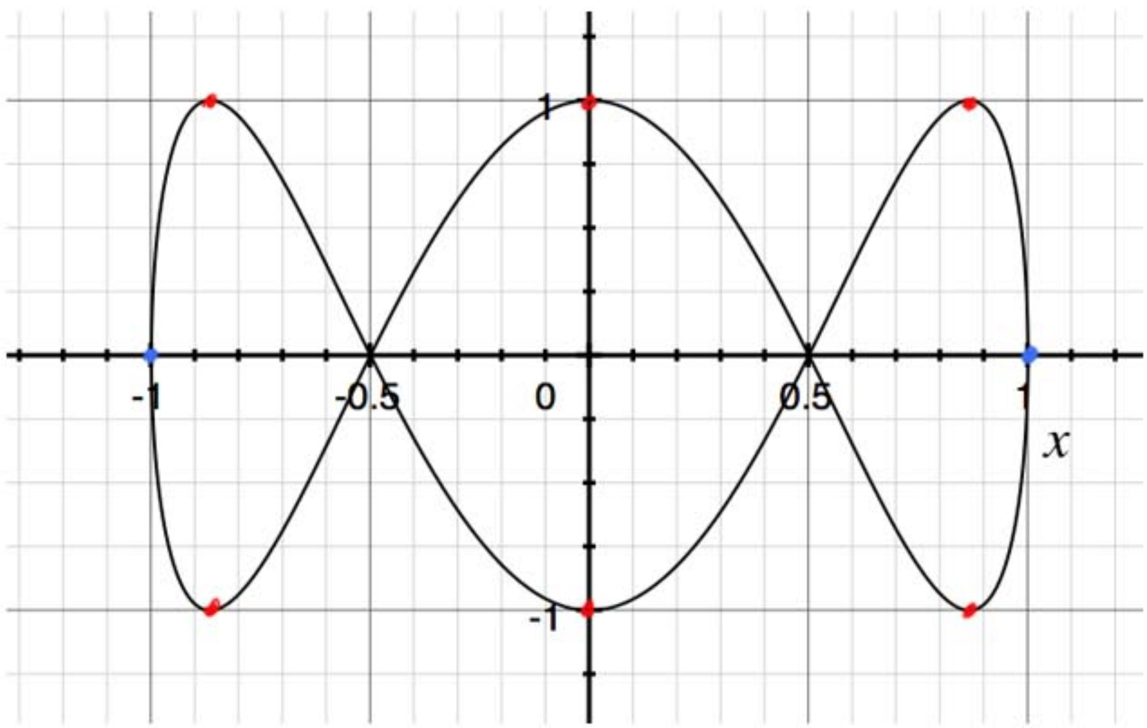
vertical : $\dfrac{-3\sin(3t)}{\cos t} = \dfrac{\text{nonzero}}{\text{zero}} \Rightarrow \cos t = 0 \quad t \in [-\pi, \pi]$

$\Rightarrow t = \pm \frac{\pi}{2}$

(notice $-3\sin(3t) \neq 0$ at these t-values)

$t = \frac{\pi}{2}$: $x = 1$, $y = 0$ $\quad (1,0)$

$t = -\frac{\pi}{2}$: $x = -1$, $y = 0$ $\quad (-1,0)$



Arc Length with Parametric Curves

If a curve C in the x-y plane is described by the parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ $\alpha \le t \le \beta$,

where f' and g' are continuous on $[\alpha, \beta]$, and C is traversed exactly once as t increases from α to β, then the length of C is

$$S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(Derivation is similar to that for $y = f(x)$, Lesson 20.)

Ex. $\begin{cases} x = e^t + e^{-t} \\ y = 5 - 2t \end{cases}$ find the arclength of curve over $0 \le t \le 3$

$$\frac{dx}{dt} = e^t + \overbrace{e^{-t}(-1)}^{\text{by chain rule}} = e^t - e^{-t}$$

$$\frac{dy}{dt} = -2 \qquad \alpha = 0 \quad \beta = 3$$

$$\text{arclength} = \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}\, dt$$

$(e^t - e^{-t})(e^t - e^{-t})$

$(e^t)^2 - \underbrace{e^t e^{-t}}_{1} - \underbrace{e^{-t} e^t}_{1} + (e^{-t})^2$

$e^{2t} - 2 + e^{-2t}$

$$= \int_0^3 \sqrt{\underbrace{e^{2t} \underbrace{-2} + e^{-2t}} \underbrace{+4}}\, dt$$

$$= \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt$$

$$= \int_0^3 \sqrt{(e^t + e^{-t})^2}\, dt = \int_0^3 (e^t + e^{-t})\, dt =$$

since

$e^t + e^{-t} > 0$

$$= \left[e^t - e^{-t} \right]_0^3$$

$\int e^{at}\, dt = \frac{1}{a} e^{at} + C$

$\int e^{-t}\, dt = \frac{1}{-1} e^{-t} + C = -e^{t} + C$

$$= \left(e^{3} - e^{-3}\right) - \left(e^{0} - e^{0}\right)$$

$$= e^{3} - e^{-3} = e^{3} - \frac{1}{e^{3}} .$$