Math 20200
Calculus II

Lesson 22
Parametric Curves

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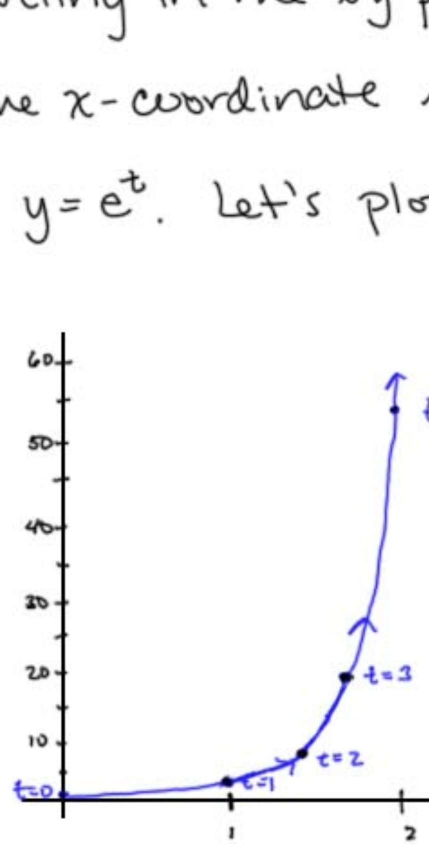
Parametric Curves

Consider a particle traveling in the x-y plane, and at any time $t \geq 0$, the x-coordinate is $x = \sqrt{t}$ and the y-coordinate is $y = e^t$. Let's plot.

t	$x = \sqrt{t}$	$y = e^t$
0	$\sqrt{0} = 0$	$e^0 = 1$
1	$\sqrt{1} = 1$	$e^1 = e \approx 2.71$
2	$\sqrt{2} \approx 1.4$	$e^2 \approx 7.4$
3	$\sqrt{3} \approx 1.7$	$e^3 \approx 20.1$
4	$\sqrt{4} = 2$	$e^4 \approx 54.6$



parametric curve (parameter t)

To find the equation of the curve in x and y (without the parameter t),

Method 1: solve for t in one equation, and sub in to the other equation

above, $x = \sqrt{t} \Rightarrow t = x^2$

$y = e^t = e^{x^2} \quad \therefore \quad y = e^{x^2}$.

Method 2: use identities (example below)

Note, the parametrized curve offers more information than the equation in x and y, because the parametrized curve tells us at what time the particle is at any given point.

Also note, if the parameter t represents time, then $t \geq 0$. But otherwise, t can span all reals, or any given interval.

Ex. Eliminate the parameter t and find the equation in x and y: $\begin{cases} x = t+2 \\ y = \sin t \end{cases}$

method 1: $x = t+2 \Rightarrow t = x-2$

$y = \sin t = \sin(x-2) \quad \therefore\ y = \sin(x-2).$

Ex. Eliminate the parameter t and find the equation in x and y: $\begin{cases} x = 2\sin t \\ y = 3\cos t \end{cases}$ then plot the curve.

method 2: we know $\sin^2 t + \cos^2 t = 1$

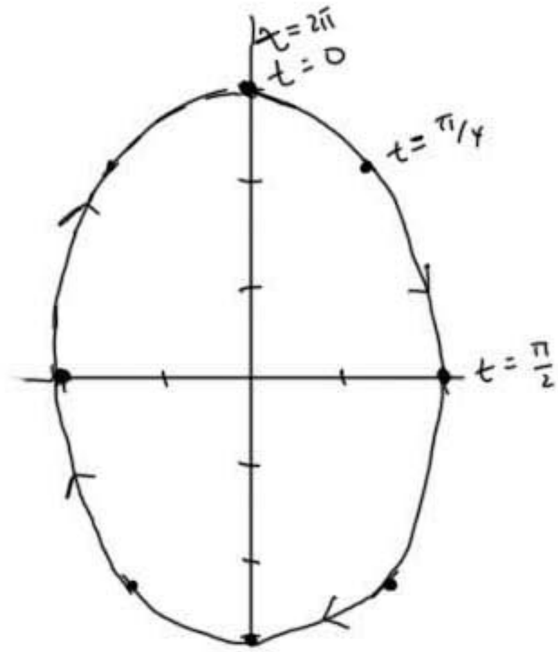
so $\frac{x}{2} = \sin t$ $\Rightarrow$ $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{y}{3} = \cos t$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ ellipse.

To plot, make a table:

t	$x = 2\sin t$	$y = 3\cos t$
0	0	3
$\frac{\pi}{4}$	$\sqrt{2} \approx 1.4$	$\frac{3\sqrt{2}}{2} \approx 2.1$
$\frac{\pi}{2}$	2	0
$\frac{3\pi}{4}$	$\sqrt{2}$	$-\frac{3\sqrt{2}}{2}$
π	0	-3
$\frac{5\pi}{4}$	$-\sqrt{2}$	$-\frac{3\sqrt{2}}{2}$
$\frac{3\pi}{2}$	-2	0
$\frac{7\pi}{4}$	$-\sqrt{2}$	$\frac{3\sqrt{2}}{2}$
2π	0	3



Ex. Match the parametric equations with the corresponding curve. Explain your work.

1. $\begin{cases} x = t^2 - 1 \\ y = t^4 \end{cases}$ $x \geq -1$, $y \geq 0$

2. $\begin{cases} x = t - 1 \\ y = t^3 \end{cases}$ $x \in \mathbb{R}$, $y \in \mathbb{R}$, $y = (x+1)^3$

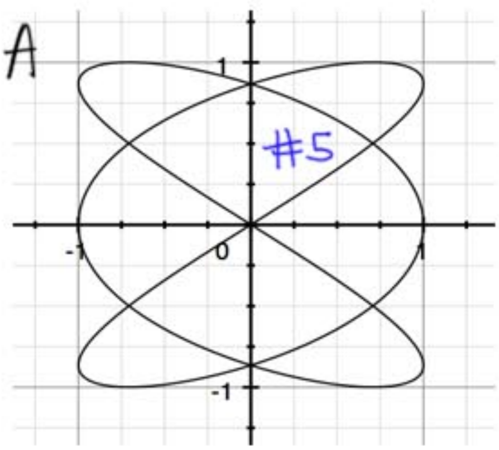
3. $\begin{cases} x = t^2 - 1 \\ y = \sin t \end{cases}$ $x \geq -1$ ← not periodic, ← is periodic, $-1 \leq y \leq 1$

4. $\begin{cases} x = t^2 - 1 \\ y = \sin \underline{\underline{2t}} \end{cases}$ similar to #3, y values changing faster here

5. $\begin{cases} x = \cos 3t \\ y = \sin 2t \end{cases}$ $-1 \leq x \leq 1$, $-1 \leq y \leq 1$

6. $\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$ $-3 \leq x \leq 3$, $-2 \leq y \leq 2$

A



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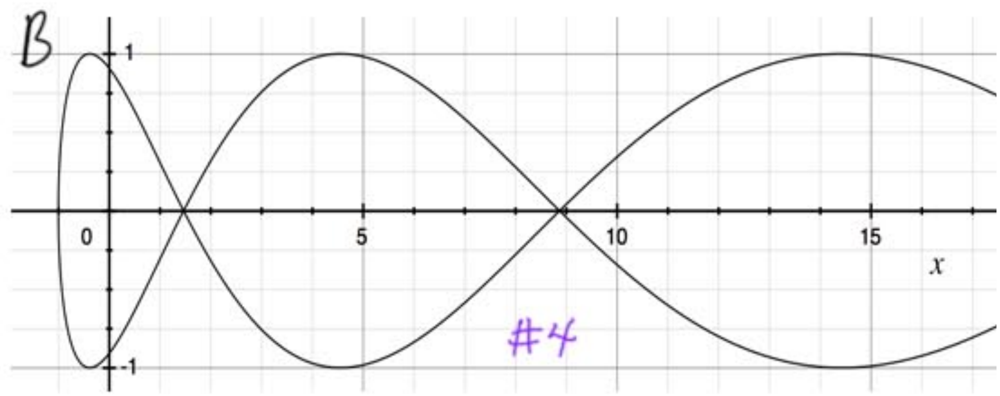

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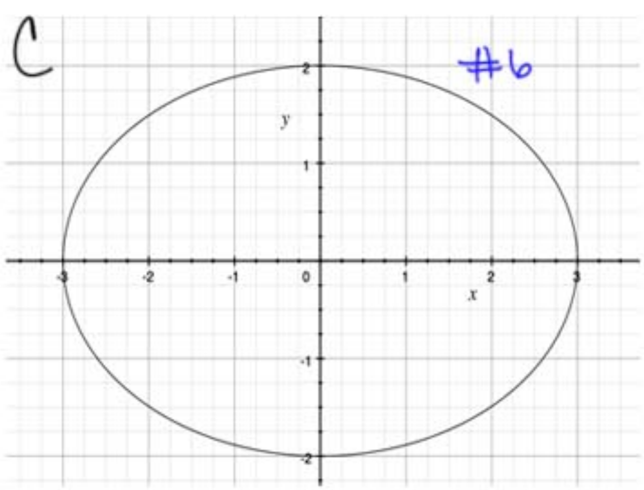

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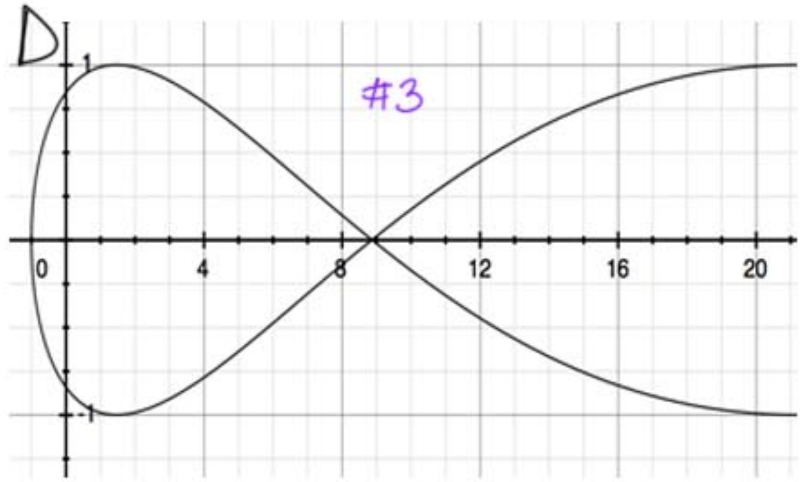

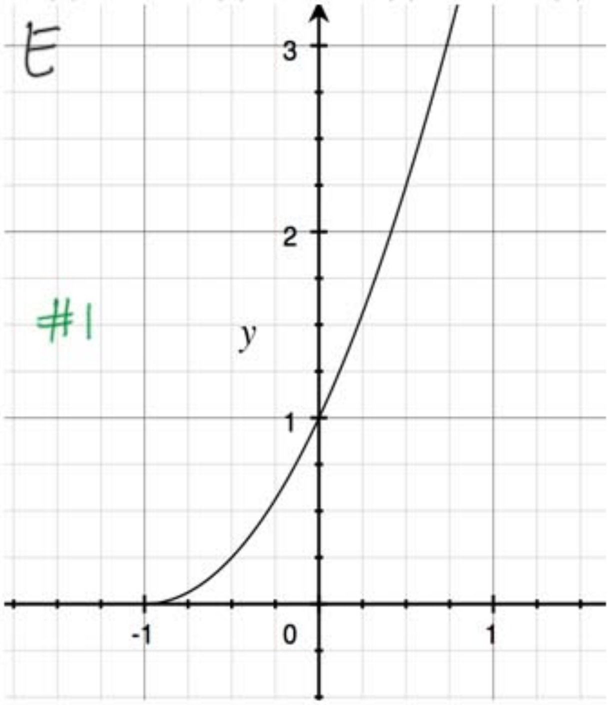

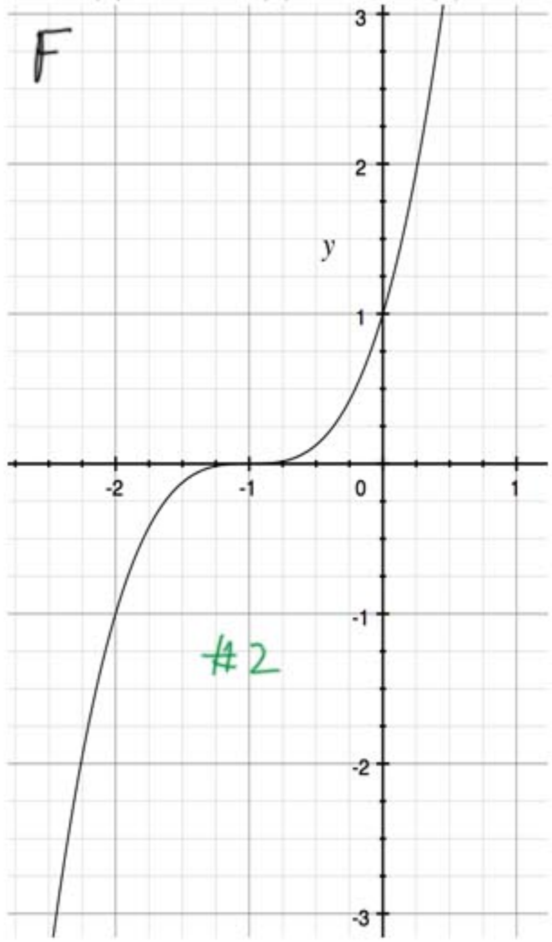

Ex. Plot the parametric curve: $\begin{cases} x = 5\sin t \\ y = t^2 \end{cases}$ $t \in [-\pi, \pi]$

Work on this problem on your own

t	$x = 5\sin t$	$y = t^2$	
$-\pi$	$5\sin(-\pi) = 0$	$(-\pi)^2 = \pi^2$	$(0, \pi^2) \approx (0, 10)$
$-\frac{\pi}{2}$	$5\sin(-\frac{\pi}{2}) = -5$	$(\frac{-\pi}{2})^2 = \frac{\pi^2}{4} \approx \frac{10}{4}$	$(-5, \frac{\pi^2}{4}) \approx (-5, \frac{10}{4})$ $2\frac{1}{2}$
0	0	0	$(0, 0)$
$\frac{\pi}{2}$	5	$\frac{\pi^2}{4} \approx \frac{10}{4}$	$(5, \frac{\pi^2}{4}) \approx (5, \frac{10}{4})$
π	0	$\pi^2 \approx 10$	$(0, \pi^2) \approx (0, 10)$