

# Math 20200

## Calculus II

### Lesson 20

### Arc Length

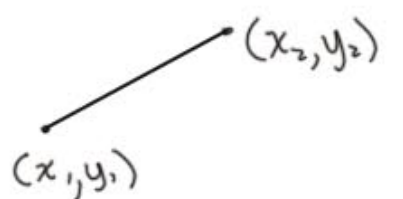
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#### Table of Contents:

1. Mean Value Theorem	02:46	p.3
2. Integral for arc length	06:05	p.4

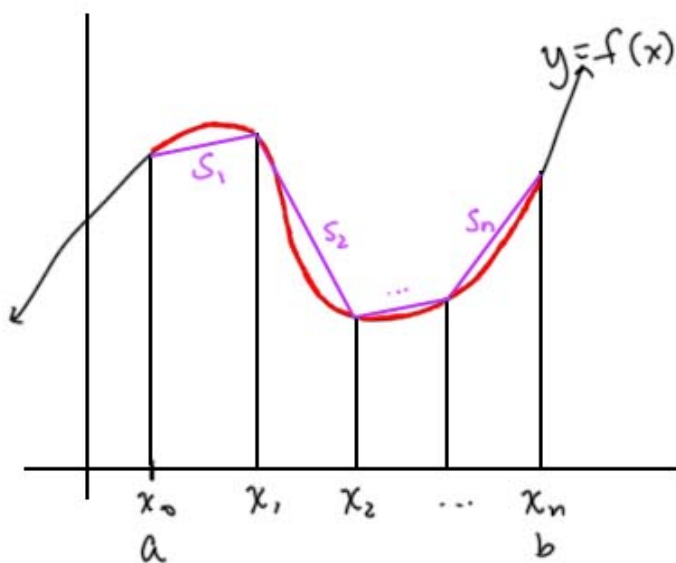
# Arc Length

So far we know how to find the length of linear segments, i.e. the distance between two points:



$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

To find the length of a curve, we approximate with linear segments, and take the limit as the number of segments approaches infinity.



$n = \#$  of segments

So consider  $f(x)$  having  $f'(x)$  continuous on  $[a, b]$ . To find the length of the graph of  $y=f(x)$  over  $a \leq x \leq b$ , we start by dividing  $[a, b]$  into  $n$  equal subintervals:

$$x_0 = a \quad \Delta x = \frac{b-a}{n} \quad \left( \begin{array}{l} \text{same as for approximating} \\ \text{definite integrals} \end{array} \right)$$
$$x_i = a + i \Delta x$$

The length of the  $i^{\text{th}}$  segment (the segment over  $[x_{i-1}, x_i]$ ) is:

$$S_i = \sqrt{\underbrace{(x_i - x_{i-1})^2}_{\Delta x} + \underbrace{(f(x_i) - f(x_{i-1}))^2}_{\text{mean value theorem}}}$$

Recall: The mean value theorem:

For  $f(x)$  continuous on  $[a, b]$

differentiable on  $(a, b)$

$$\exists c \in (a, b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\text{or } f(b) - f(a) = f'(c)(b - a)$$

$$\therefore \exists x_i^* \in (x_{i-1}, x_i) \Rightarrow$$

$$S_i = \sqrt{(\Delta x)^2 + (f'(x_i^*) \underbrace{(x_i - x_{i-1})}_{\Delta x})^2}$$

$$= \sqrt{(\Delta x)^2 (1 + (f'(x_i^*))^2)}$$

$$= \Delta x \sqrt{1 + (f'(x_i^*))^2} \quad \text{since } \Delta x > 0.$$

and the length of the curve, called arc length is

$$S = \lim_{n \rightarrow \infty} \sum_{i=1}^n S_i$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x \sqrt{1 + (f'(x_i^*))^2}$$

$$\therefore S = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Similarly for  $x = g(y)$  with  $g'(y)$  continuous on  $[c, d]$ .

$$S = \int_c^d \sqrt{1 + (g'(y))^2} dy.$$

Ex. Find The length of The curve :

$$f(x) = \ln(\cos x) \quad 0 \leq x \leq \frac{\pi}{3}.$$

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

$$\text{arclength} = \int_0^{\pi/3} \sqrt{1 + (-\tan x)^2} dx$$

$$= \int_0^{\pi/3} \sqrt{1 + \tan^2 x} dx = \int_0^{\pi/3} \sqrt{\sec^2 x} dx$$

$$= \int_0^{\pi/3} \sec x dx \quad \text{since } \sec x \geq 0 \text{ on } [0, \pi/3]$$

$$= \ln |\sec x + \tan x| \Big|_0^{\pi/3} =$$

$$= \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \ln |\sec 0 + \tan 0|$$

$$= \ln |2 + \sqrt{3}| - \ln |1 + 0|$$

$$= \ln(2 + \sqrt{3}).$$

Ex. Find The length of The curve :

$$f(x) = \frac{x^3}{6} + \frac{1}{2x} \quad 1 \leq x \leq 2.$$

$$f'(x) = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

$$= \frac{x^2}{2} - \frac{1}{2x^2}$$

$$\text{arclength} = \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} dx$$

$$= \int_1^2 \sqrt{\underbrace{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}_{+\frac{1}{2}}} dx$$

$$\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)$$

$$= \frac{x^4}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4x^4}$$

$$= \int_1^2 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} dx$$

$$= \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$$

$$= \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} dx$$

notice  $\frac{x^2}{2} + \frac{1}{2x^2} > 0$   
on  $[1, 2]$

$$= \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$

$$= \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2}x^{-2}\right) dx = \left[\frac{x^3}{6} + \left(\frac{1}{2}\right)\frac{x^{-1}}{(-1)}\right]_1^2$$

$$= \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2 = \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right)$$

$$= \frac{7}{6} - \frac{1}{4} + \frac{1}{2} = \frac{14}{12} - \frac{3}{12} + \frac{6}{12} = \frac{17}{12}$$

Ex.  $y^2 = 4x$   $0 \leq y \leq 2$  find arclength

$$\Rightarrow x = g(y)$$

$$x = \frac{1}{4}y^2$$



Work on this problem  
on your own

$$g'(y) = \frac{1}{2}y$$

$$\text{arclength} = \int_0^2 \sqrt{1 + \left(\frac{1}{2}y\right)^2} dy$$

$$\sqrt{a^2 + x^2}$$
$$x = a \tan \theta$$

trig sub.

$$= \int_0^2 \sqrt{1 + \frac{1}{4}y^2} dy = \int_0^2 \sqrt{\frac{1}{4}(4 + y^2)} dy$$

$$= \frac{1}{2} \int_0^2 \sqrt{4 + y^2} dy$$

$$y = 2 \tan \theta$$
$$dy = 2 \sec^2 \theta d\theta$$



$$= \frac{1}{2} \int_0^{\pi/4} \sqrt{4+4\tan^2\theta} \cdot \cancel{2} \sec^2\theta d\theta$$

$$y = 2 \tan\theta$$

$$0 = 2 \tan\theta$$

$$\theta = 0$$

$$= \int_0^{\pi/4} \sqrt{4(1+\tan^2\theta)} \sec^2\theta d\theta =$$

$$2 = 2 \tan\theta$$

$$\theta = \frac{\pi}{4}$$

$$= 2 \int_0^{\pi/4} \sqrt{1+\tan^2\theta} \sec^2\theta d\theta$$

$\sec^2\theta$  and  $\sec\theta > 0$  on  $[0, \pi/4]$

$$= 2 \int_0^{\pi/4} \sec^3\theta d\theta \quad \text{secant reduction } n=3$$

$$\int \sec^n x dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx$$

$$= 2 \left[ \frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \int \sec\theta d\theta \right]_0^{\pi/4}$$

$$= 2 \left[ \frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \ln|\sec\theta + \tan\theta| \right]_0^{\pi/4}$$

$$= \left( \sec^{\pi/4} \tan^{\pi/4} + \ln|\sec^{\pi/4} + \tan^{\pi/4}| \right)$$

$$- (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$$

$$= (\sqrt{2}(1) + \ln |\sqrt{2} + 1|) - (0 + \ln(1))$$

$$= \sqrt{2} + \ln(\sqrt{2} + 1).$$

Ex. Find The circumference of the circle of radius  $r$  by doubling the arc length of

the semi-circle:  $y = \sqrt{r^2 - x^2}$   $[-r, r]$

$$y' = \frac{1}{2} (r^2 - x^2)^{-1/2} (-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$C = 2 \int_{-r}^r \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} dx$$

$$= 2 \int_{-r}^r \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx$$

$$= 2 \int_{-r}^r \sqrt{\frac{r^2 - \cancel{x^2} + \cancel{x^2}}{r^2 - x^2}} dx$$

$$= 2 \int_{-r}^r \sqrt{\frac{r^2}{r^2 - x^2}} dx = 2 \int_{-r}^r \frac{r}{\sqrt{r^2 - x^2}} dx$$

$$= 2 \int_{-r}^r \frac{r}{\sqrt{r^2(1 - \frac{x^2}{r^2})}} dx = 2 \int_{-r}^r \frac{\cancel{r}}{\cancel{r} \sqrt{1 - (\frac{x}{r})^2}} dx$$

$$= 2 \int_{-r}^r \frac{1}{\sqrt{1 - (\frac{x}{r})^2}} dx \quad u = \frac{x}{r} \quad du = \frac{1}{r} dx$$

$x = -r \quad x = r$   
 $u = -1 \quad u = 1$

$$= 2r \int_{-r}^r \frac{\frac{1}{r} dx}{\sqrt{1 - (\frac{x}{r})^2}} = 2r \int_{-1}^1 \frac{du}{\sqrt{1 - u^2}}$$

$$= 2r \arcsin u \Big|_{-1}^1 = 2r (\arcsin 1 - \arcsin(-1))$$
$$= 2r \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) = 2\pi r.$$