Math 20200
Calculus II

Lesson 19
Volumes by Cylindrical Shells

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Volumes by Cylindrical Shells

(for volumes of solids of revolution)

In lesson 18, we learned volumes by slicing:

$$\int_a^b A(x)\,dx \quad \text{or} \quad \int_c^d A(y)\,dy$$

when used with volumes of revolution,

$$A(x) = \pi\,(R(x))^2$$
$$A(y) = \pi\,(R(y))^2$$

Now we compute volumes by the method of cylindrical shells   (different method!)

Recall That for The solids of revolution in Lesson 18, The disks or washers were obtained by taking a cut of The region that was perpendicular to The axis of revolution, and following that cut through The revolution:

$y = \sqrt{x}$

$r = \sqrt{x}$

disks

Lesson 18
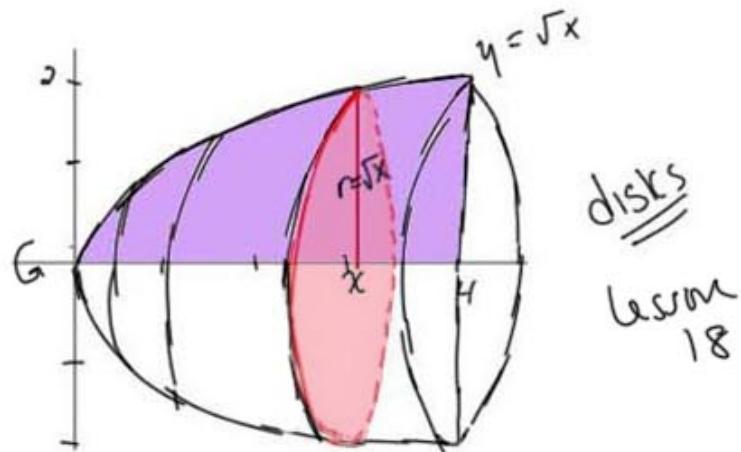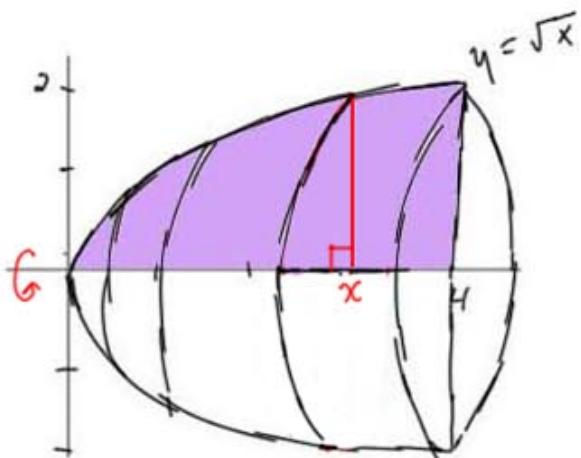
Now, we take a cut of The region that is parallel to The axis of revolution, and follow it through The revolution:

$y = \sqrt{x}$
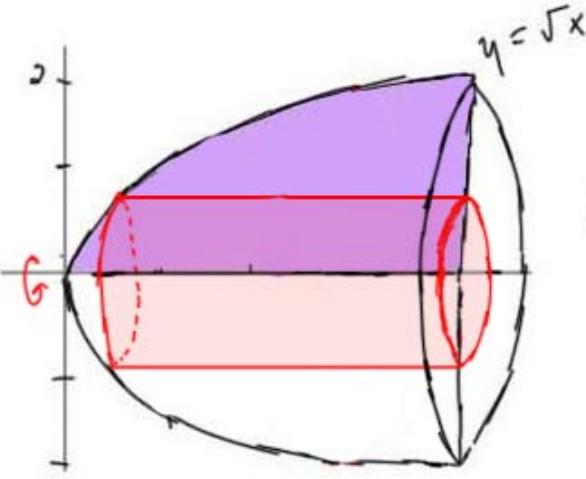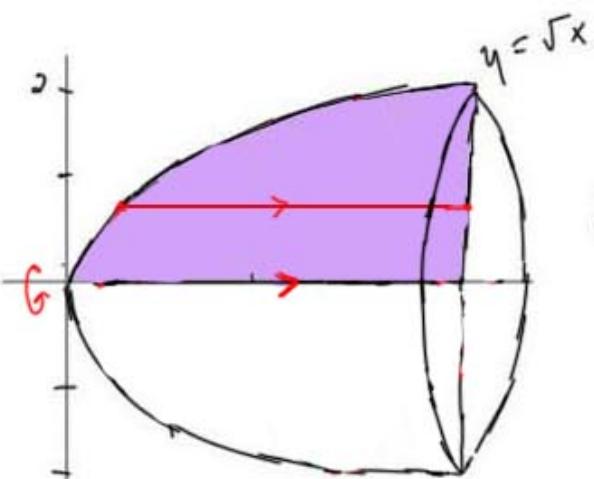
volume of The cylindrical shell (hollow cylinder):

Volume of The shell $\approx$

$\underbrace{\text{Surface area}}_{\substack{\text{Circumference of}\\ \text{Circle} \cdot \text{height}}} \cdot \underbrace{\text{Thickness of}}_{\substack{\text{shell}\\ \Delta x\\ \text{or}\\ \Delta y}}$

unroll The shell:

area =
$2\pi r \cdot h$

h

$2\pi r$
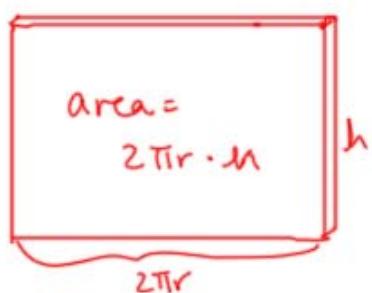
$= 2\pi r(x) h(x) \Delta x$

$OR = 2\pi r(y) h(y) \Delta y$

then volume of The solid $= \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} 2\pi r(x_i) h(x_i) \Delta x$

where $n = \#$ shells

$= \int_{a}^{b} 2\pi r(x) h(x) \, dx$

$OR = \int_{c}^{d} 2\pi r(y) h(y) \, dy$

∴ volume of the above solid :

$x = y^2$
$y = \sqrt{x}$
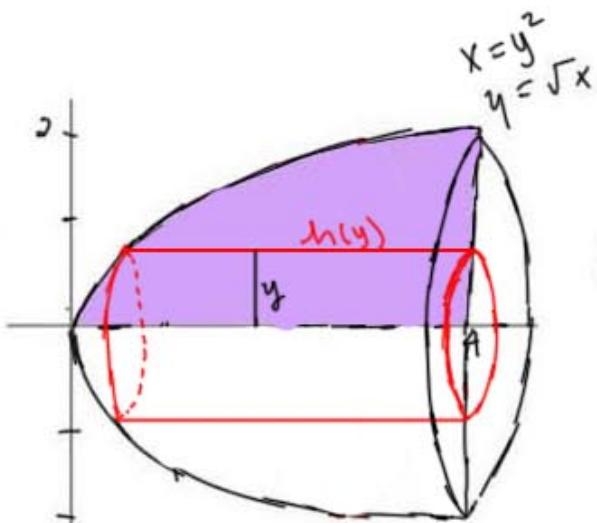
thickness of shell is in the y-direction, so Δy

radius $r(y) = y$

$h(y) = \text{right} - \text{left}$

$= \underset{\underset{x=4}{\uparrow}}{4} - \underset{\underset{x=y^2}{\uparrow}}{y^2}$

and notice $\quad 0 \le y \le 2$

∴ volume of solid $= \displaystyle\int_0^2 2\pi \, r(y) \, h(y) \, dy$

$= \displaystyle\int_0^2 2\pi \, y \, (4 - y^2) \, dy$

$= 2\pi \displaystyle\int_0^2 (4y - y^3) \, dy = 2\pi \left[2y^2 - \dfrac{y^4}{4} \right]_0^2 =$

$$= 2\pi \left[\left(2 \cdot 4 - \frac{16}{4} \right) - (0 - 0) \right] = 2\pi (8 - 4) = 8\pi.$$

(same answer as
in lesson 18)

Ex. Find the volume of the solid formed by
revolving the region about the y axis.
The region is bounded by $y = -\cos x$
and the x-axis over $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$.

Start by sketching:

notice that we
can't make a cut of
this region that is
perpendicular to the
axis of revolution:

$y = -\cos x$

same curve
left and right

$y = -\cos x$
we can't
describe
this length
as a function
of y

(can't solve for x)
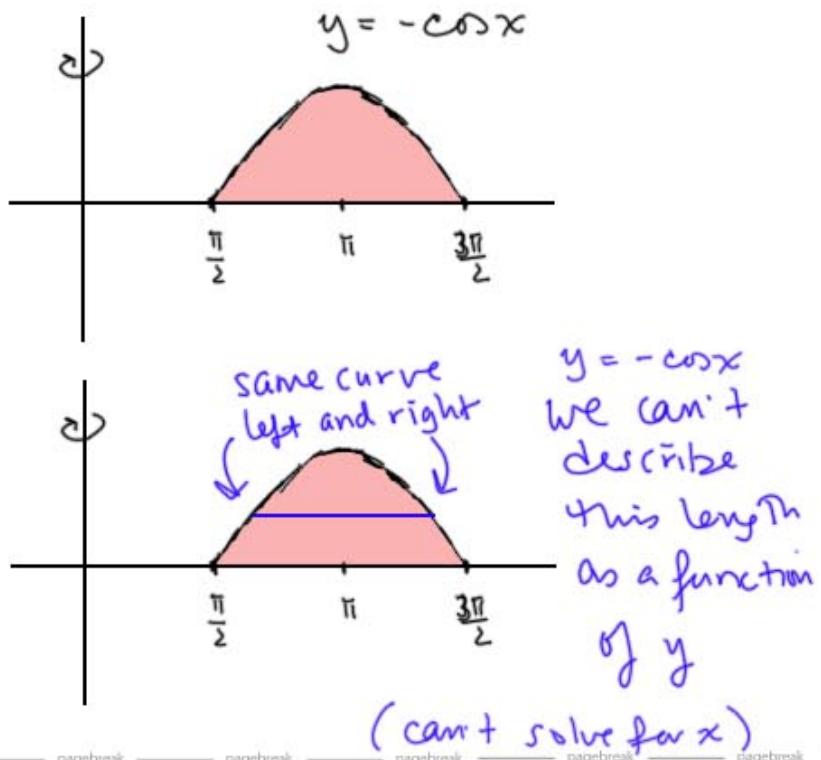
We can cut the region parallel to the axis of revolution:

$h(x) = $ top $-$ bottom
$= -\cos x - 0$
$= -\cos x$
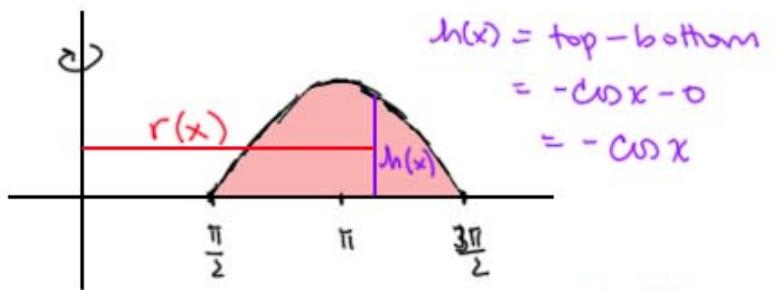
r(x) is the distance from the cut to the axis of revolution

solid of revolution:

like a jelly ring:
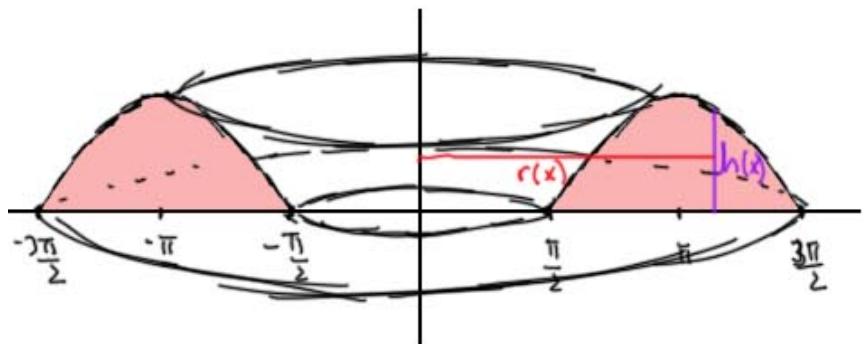
cylindrical shell:

Cylindrical shell:
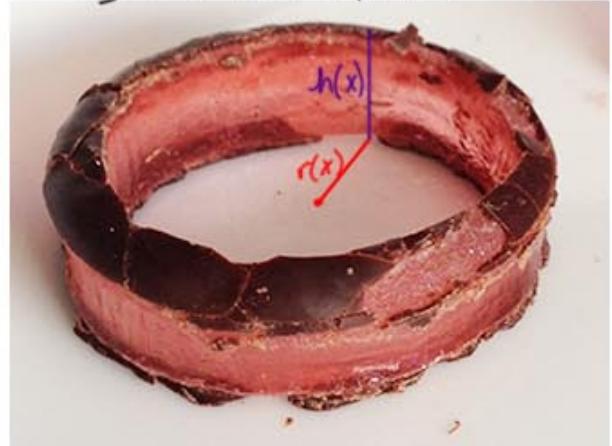

and the volume $= \displaystyle\int_{\pi/2}^{3\pi/2} 2\pi\, r(x)\, h(x)\, dx = \int_{\pi/2}^{3\pi/2} 2\pi\, x\,(-\cos x)\, dx$

$r(x) = $ right $-$ left
$= x - 0$
$= x$

$= \ldots$ (by parts) $= 4\pi^2.$

Ex. Same region as above, revolved around

$x = \pi/4$:

$h(x) = $ top $-$ bottom
$= -\cos x - 0$
$= -\cos x$
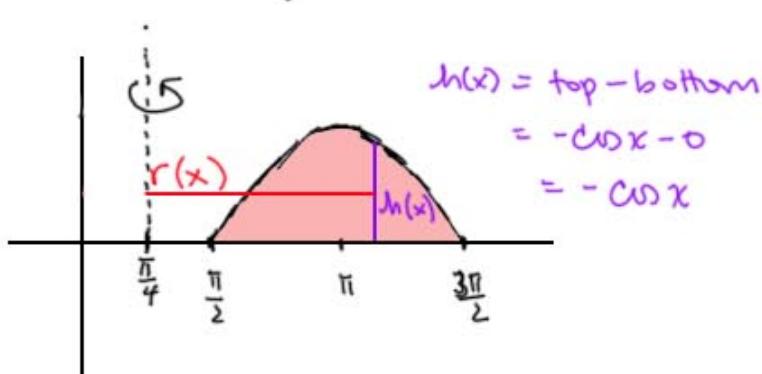
$r(x)$ is the distance from the cut to the axis of revolution

$r(x) = $ right $-$ left
$= x - \frac{\pi}{4}$
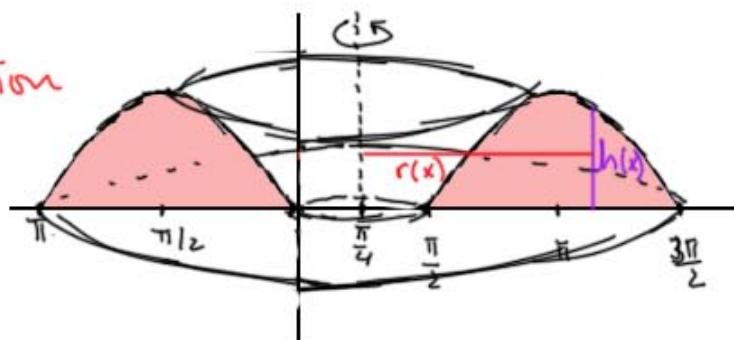
$$\text{Volume} = \int_{\pi/2}^{3\pi/2} 2\pi \, r(x) \, h(x) \, dx = \int_{\pi/2}^{3\pi/2} 2\pi \left(x - \frac{\pi}{4}\right)(-\cos x) \, dx$$

$$= \dots \text{(by parts)} = 3\pi^2.$$

Ex. Find the volume of the solid generated by revolving the region bounded by $y = x^2$, $y = 0$, $x = -1$, $x = 1$

around the line $x = -2$.
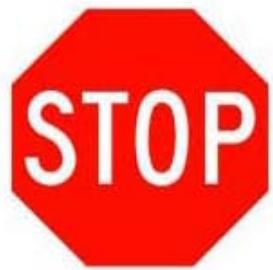
STOP Work on this problem on your own

$$r(x) = \overset{right}{x} - \overset{left}{(-2)}$$

$$= x + 2$$
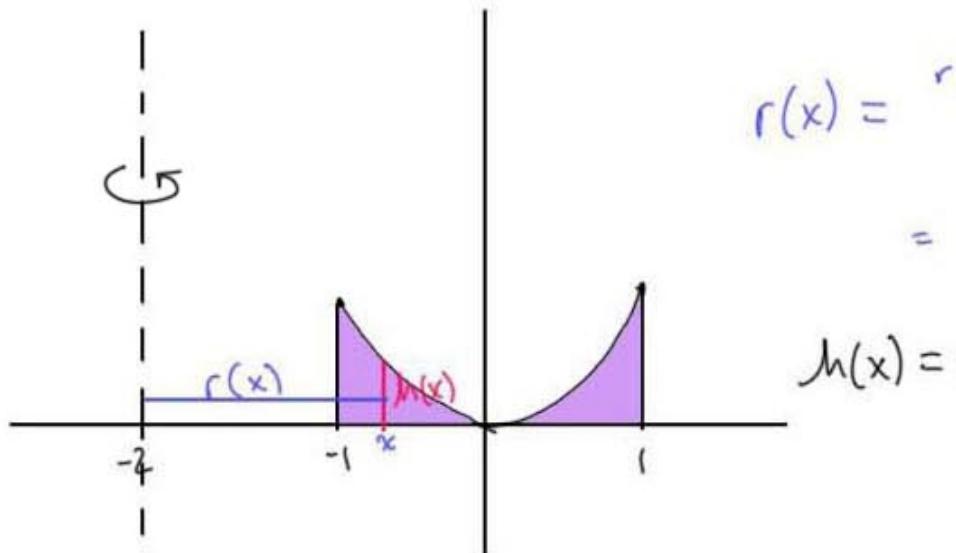
$$h(x) = x^2$$

$$Volume = \int_{-1}^{1} 2\pi \, r(x)\, h(x)\, dx = \int_{-1}^{1} 2\pi (x+2) x^2\, dx$$

$$= \ldots = \frac{8\pi}{3}.$$
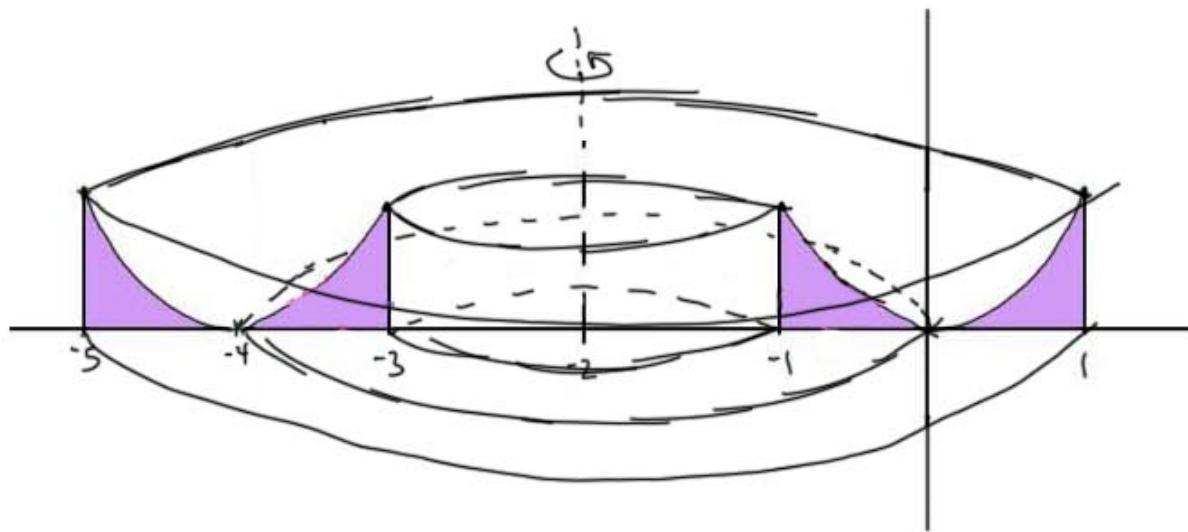
Ex. Find the volume of the solid generated by revolving the region bounded by $x = y^2$ and $x = 4$ around $y = 2$.
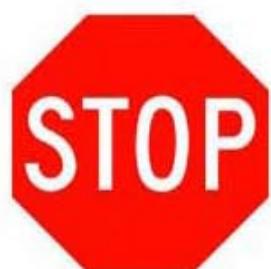
STOP Work on this problem on your own
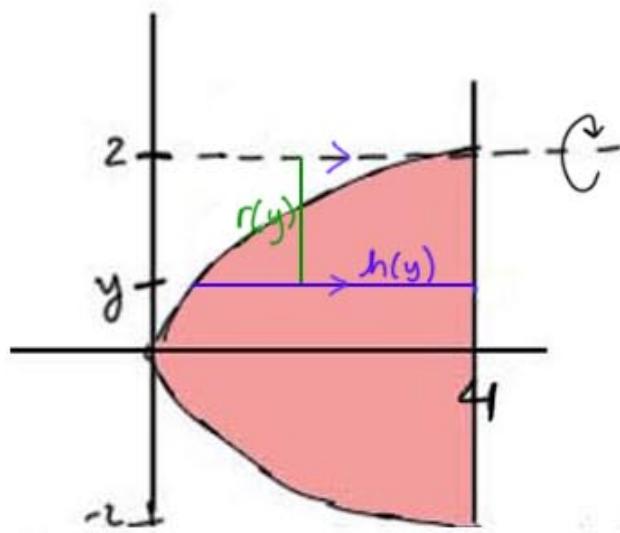
Cutting the region parallel to the axis of revolution uses cylindrical shells.
$r(y)$ is the distance from the cut to the axis of revolution
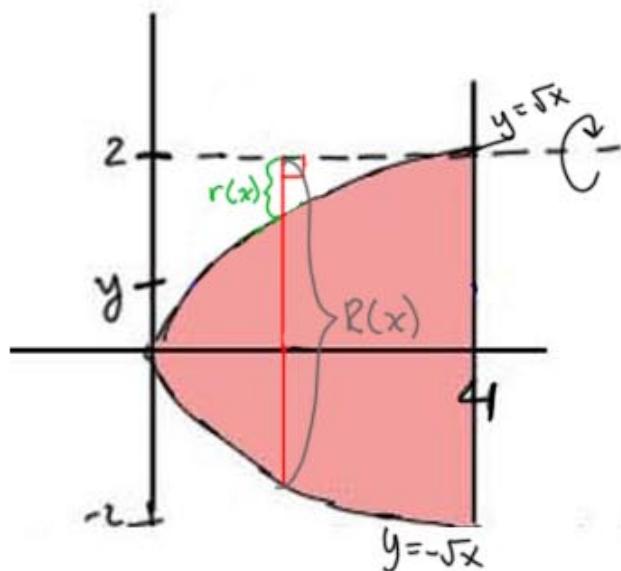
Cutting the region perpendicular to the axis of revolution uses washers (lesson 18)

$h(y)$ = right – left

$\quad = 4 - y^2$

$\qquad \underset{x=4}{\uparrow} \quad \underset{x=y^2}{\uparrow}$

$r(y)$ = top – bottom

$\qquad 2 - y$

$$V = \int_{-2}^{2} 2\pi \, r(y) \, h(y) \, dy$$

$$= \int_{-2}^{2} 2\pi \, (2-y)(4-y^2) \, dy$$

$$= \ldots \quad = \frac{128\pi}{3}$$

$R(x)$ = top – bottom

$\quad = 2 - (-\sqrt{x})$

$\qquad \underset{y=2}{\uparrow} \qquad \underset{y=-\sqrt{x}}{\uparrow}$

$\quad = 2 + \sqrt{x}$

$r(x)$ = top – bottom

$\quad = 2 - \sqrt{x}$

$$V = \int_{a}^{b} \pi \left((R(x))^2 - (r(x))^2 \right) dx$$

$$= \int_{0}^{4} \pi \left((2+\sqrt{x})^2 - (2-\sqrt{x})^2 \right) dx$$

$$= \ldots \quad = \frac{128\pi}{3}.$$

∴ How to find the volume of a solid of revolution:

1) Sketch the region and identify the axis of revolution

2) determine if you want to cut the region perpendicular to the axis of revolution (disks/washers) or parallel to the axis of revolution (cylindrical shells)

ie, does your region have top and bottom curves or right and left curves?

top and bottom curves => dx integral

left and right curves => dy integral

3) if disks, find $r(x)$ or $r(y)$

if washers, find $R(x)$ and $r(x)$ OR $R(y)$ and $r(y)$

if shells, find $h(x)$ and $r(x)$ OR $h(y)$ and $r(y)$

4) integrate!