Math 20200
Calculus II

Lesson 18
Volumes by Slicing, Including Disks and Washers

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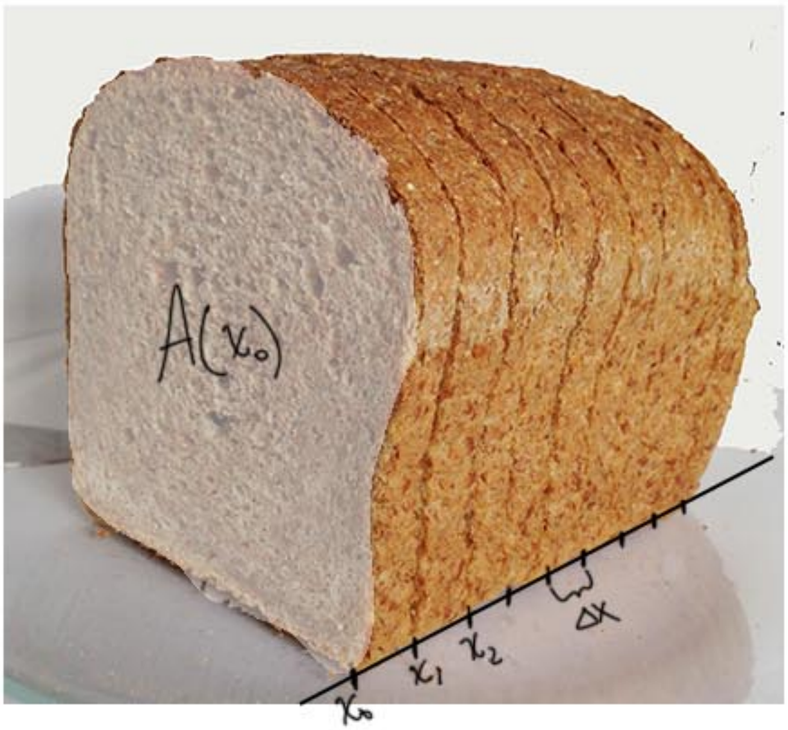
Volumes by Slicing



how can we find the volume of the loaf of bread?

make slices of uniform thickness Δx

approximate the volume of each slice

volume of i^{th} slice $= \underbrace{\text{area of face of } i^{th} \text{ slice}}_{A(x_i)} \cdot \Delta x$

total volume $\approx \sum_{i=1}^{n} A(x_i)\,\Delta x$

more slices, better approximation so let $n \to \infty$

$$\text{total volume} = \int_a^b \underbrace{A(x)}_{\text{Area of the face of the slice at any x-value}} dx$$

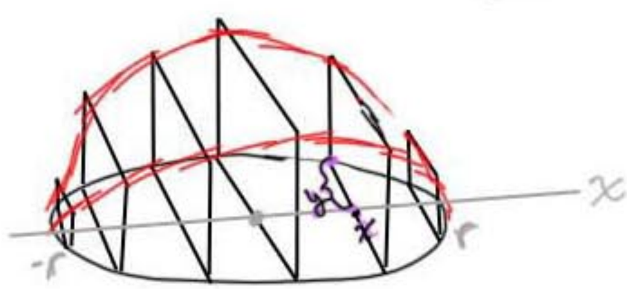
also called "cross-section"

(depends on x-value)

Volume by slicing

Ex. Find the volume of the described solid S. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are squares.

$$\text{volume} = \int_a^b \underbrace{A(x)}_{\text{area of the (square) cross-section that is perpendicular to the x-axis}} \overbrace{dx}^{\text{thickness of the slice}}$$



circle: $x^2+y^2=r^2$

$y=\sqrt{r^2-x^2}$

side of the square $2\sqrt{r^2-x^2}$

$$\begin{aligned}\text{Area of square} &= (\text{side})^2 \\ &= \left(2\sqrt{r^2-x^2}\right)^2 \\ &= 4(r^2-x^2) = A(x)\end{aligned}$$

$$\text{volume} = \int_{-r}^{r} 4(r^2 - x^2)\,dx = 4\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= 4\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$$

$$= 4\left(2r^3 - \frac{2r^3}{3}\right) = 4 \cdot \frac{4r^3}{3} = \frac{16r^3}{3}.$$

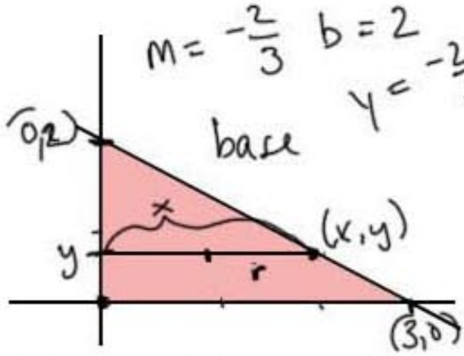
Ex. The base of S is the triangular region with vertices $(0,0)$, $(3,0)$, + $(0,2)$

Cross sections perpendicular to the y-axis are semicircles. Find the volume of S.

Work on this problem on your own

$m = -\frac{2}{3}$ $b = 2$

$y = -\frac{2}{3}x + 2$



cross sections along y, ie perpendicular to the y axis

$$\text{volume} = \int_a^b A(x)\,dx \quad \text{OR} = \int_c^d A(y)\,dy$$

area of semicircle $= \frac{1}{2}\pi r^2$ and $r = \frac{1}{2}x$ need in terms of y.

$$A(y) = \frac{1}{2}\pi\left(r(y)\right)^2$$

$$y = -\frac{2}{3}x + 2$$

$$y - 2 = -\frac{2}{3}x$$

$$x = -\frac{3}{2}(y-2)$$

$$\therefore \; r = -\frac{3}{4}(y-2). \quad \left(r = \frac{1}{2}x\right)$$

$$A(y) = \frac{1}{2}\pi\left(-\frac{3}{4}(y-2)\right)^2 = \frac{9}{32}\pi(y-2)^2$$

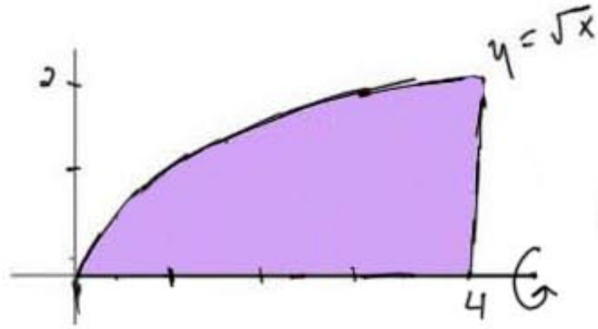
$$\text{volume} = \int_0^2 \frac{9}{32}\pi(y-2)^2\,dy = \frac{9}{32}\pi\int_0^2 (y-2)^2\,dy$$

$$= \frac{9}{32}\pi\left.\frac{(y-2)^3}{3}\right|_0^2 = \frac{9}{32}\pi\left[\frac{0^3}{3} - \frac{(-2)^3}{3}\right]$$

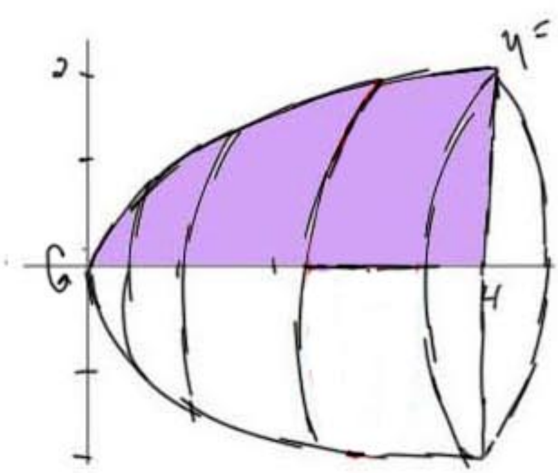
$$= \frac{9}{32}\pi\left(\frac{8}{3}\right) = \frac{3\pi}{4}.$$

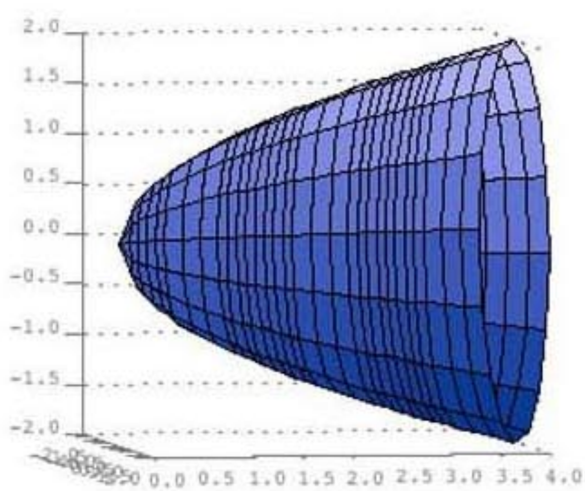
Volume of Solids of Revolution

Consider the region bounded by $y = \sqrt{x}$ and the x axis over $0 \le x \le 4$.

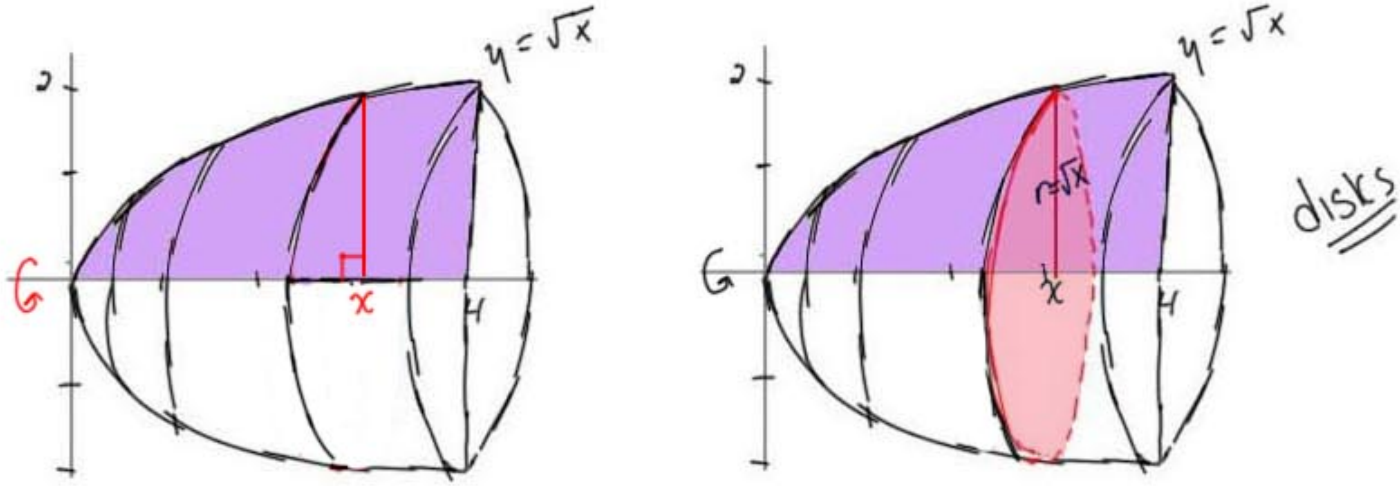


Rotating around the x axis:





We find the volume by considering slices of this solid. The slices are formed by making a cut in the region that is perpendicular to the axis of revolution, and following that through the revolution:



$A(x)$ = area at any x-value $= \pi\left(r(x)\right)^2$

Because this is a solid of revolution, any slice will be a circle (cylinder if you consider the thickness dx)

so we need to identify a function $r(x)$ for radius of the slice at any x-value.

radius $= \sqrt{x}$ Area $= A(x) = \pi\left(r(x)\right)^2$

$$= \pi\left(\sqrt{x}\right)^2 = \pi x$$

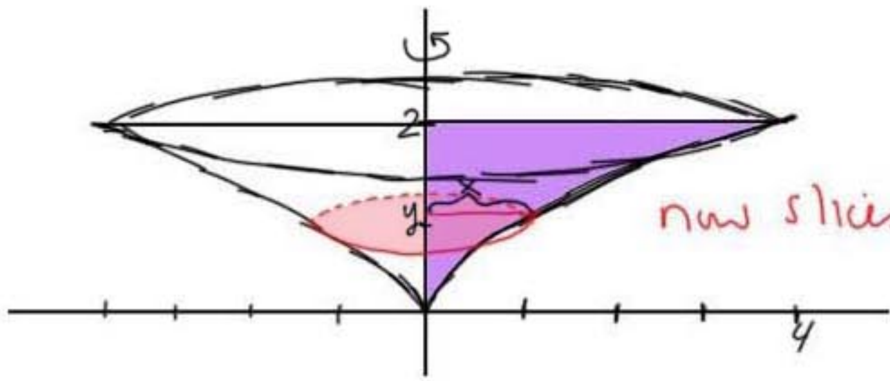
$$\text{volume of solid} = \int_0^4 A(x)\,dx = \int_0^4 \pi x \, dx =$$

$$\pi \frac{x^2}{2}\Big]_0^4 = \frac{16}{2}\pi = 8\pi = \text{volume of solid.}$$

Ex. Consider the region bounded by $y=\sqrt{x}$, $x=0$, and $y=2$.

Find the volume of the solid generated by revolving the region around the y-axis

Work on this problem on your own



disks

now slices are horizontal

area function

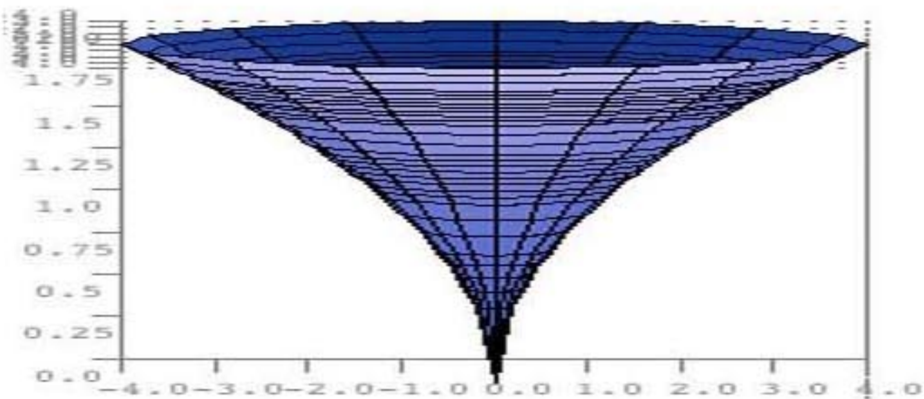
$A(y) = \pi \left(r(y)\right)^2$

need $r(y)$

$y=\sqrt{x}$

radius $= x = y^2$

$r(y) = y^2$



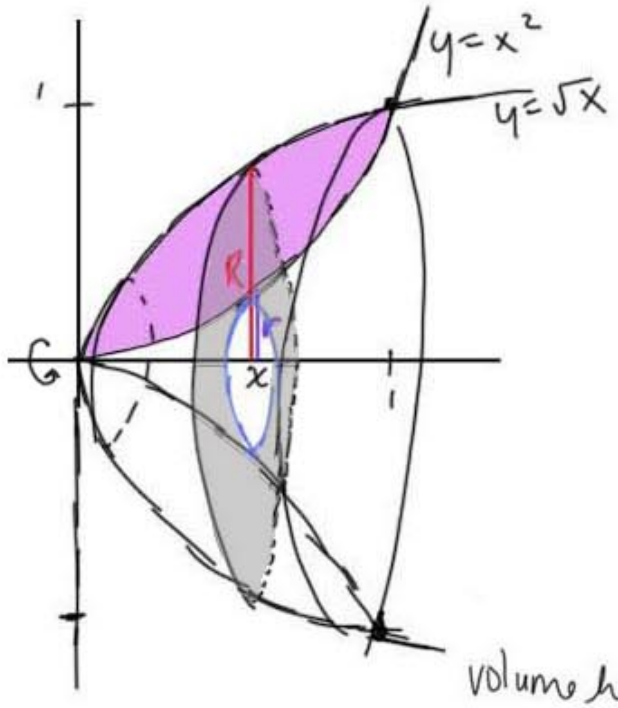
$$\text{volume} = \int_0^2 \pi \left(y^2\right)^2 \, dy = \pi \int_0^2 y^4 \, dy =$$

$$= \pi \left.\frac{y^5}{5}\right]_0^2 = \frac{32}{5}\pi .$$

Ex. Consider the region bounded by

$y=x^2$ + $y=\sqrt{x}$.

Find The volume of The solid generated by revolving The region around The x axis.



Consider subtracting volumes.

larger volume: $\int_0^1 \pi (R(x))^2 dx$

Smaller volume = $\int_0^1 \pi (r(x))^2 dx$

volume here = $\pi \int_0^1 ((R(x))^2 - (r(x))^2)\, dx \neq \pi \int_0^1 \cancel{(R-r)^2}\, dx$

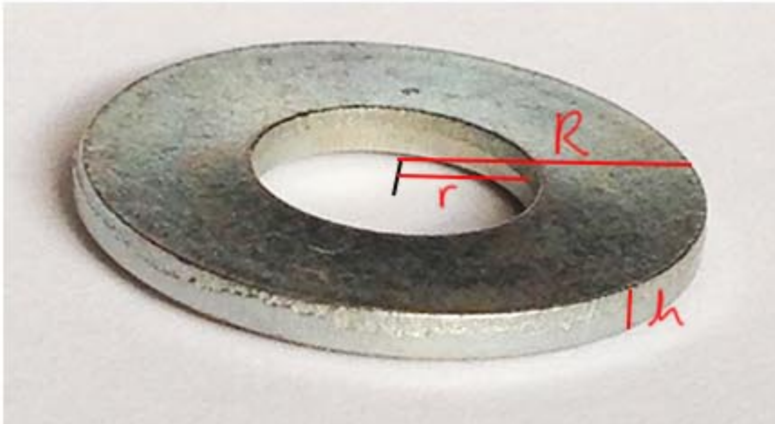
$R(x) = \sqrt{x} \qquad r(x) = x^2$

$$\text{volume} = \pi \int_0^1 \left((\sqrt{x})^2 - (x^2)^2 \right) dx = \pi \int_0^1 (x - x^4)\, dx$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \pi \frac{3}{10} = \frac{3\pi}{10}.$$

This is The method of washers.

Washer:



Volume:

$\pi R^2 h - \pi r^2 h$

$= \pi (R^2 - r^2) h$

Method of washers:

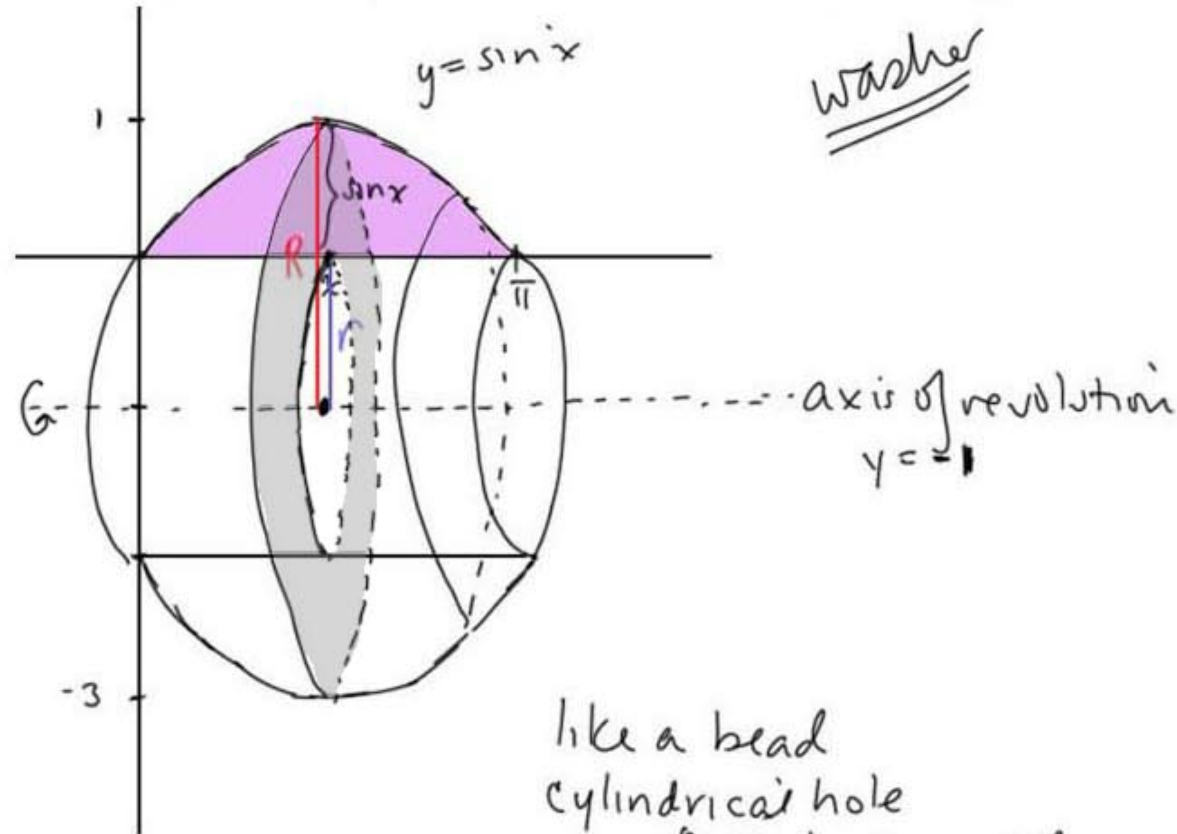
$$\text{Volume} = \pi \int_a^b \left((R(x))^2 - (r(x))^2 \right) dx$$

$$\text{OR} = \pi \int_c^d \left((R(y))^2 - (r(y))^2 \right) dy$$

Ex. Consider the region bounded by $y = \sin x$ and the x axis over the interval $0 \le x \le \pi$.

Find the volume of the solid generated by revolving the region around $y = -1$.

Work on this problem on your own



washer

like a bead
cylindrical hole
through the solid

$R(x) = \sin x + 1$ $r(x) = 1$

$\sin x - (-1)$ $= 0 - (-1)$

(top) − (bottom)   top − bottom

$$\text{volume} = \pi \int_0^{\pi} \left((R(x))^2 - (r(x))^2 \right) dx = \pi \int_0^{\pi} \left((\sin x + 1)^2 - 1^2 \right) dx$$

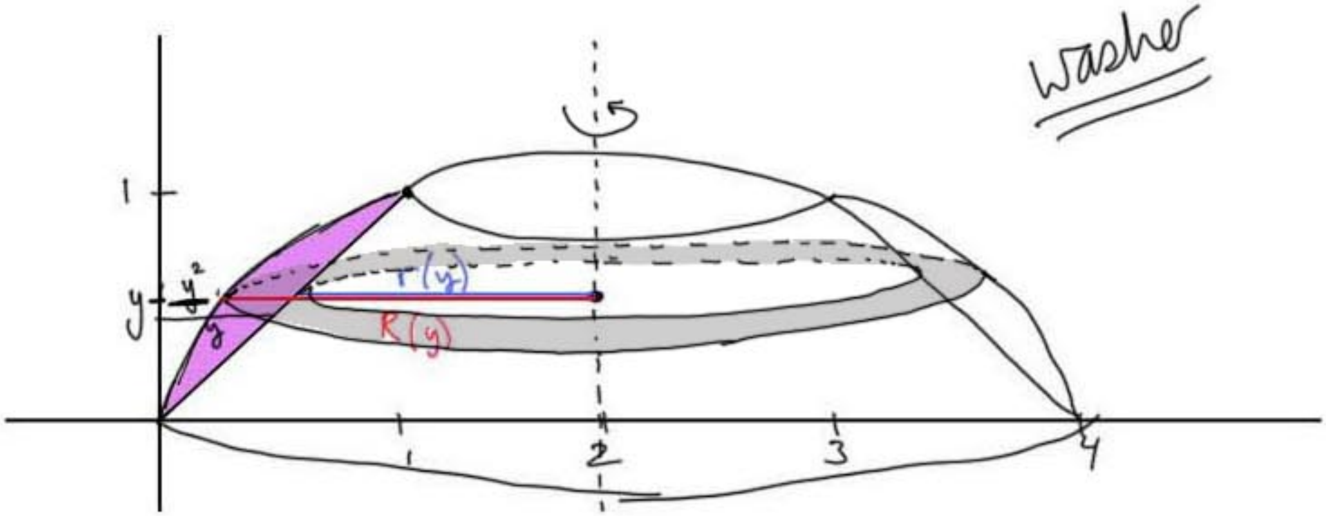
$$= \dots = \frac{\pi(\pi + 8)}{2}.$$

Ex. Find the volume of the solid generated by revolving the region around $x = 2$.

The region is bounded by $y = x$ and $y = \sqrt{x}$.

Work on this problem on your own

washer



points of intersection:

$$\left.\begin{aligned} y &= \sqrt{x} \\ y &= x \end{aligned}\right\} \text{set} = \qquad \sqrt{x} = x$$

$$x = x^2$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$x = 0$

$x = 1$

$$R(y) = \overset{\text{right - left}}{2 - y^2} \qquad r(y) = 2 - y$$

$$\underset{x=2}{\uparrow} \quad \underset{x=y^2}{\uparrow} \qquad\qquad \underset{x=2}{\uparrow} \quad \underset{x=y}{\uparrow}$$

$$\text{volume} = \pi \int_c^d \left((R(y))^2 - (r(y))^2 \right) dy$$

$$= \pi \int_0^1 \left((2-y^2)^2 - (2-y)^2 \right) dy$$

$$= \quad \ldots \quad = \frac{8\pi}{15}.$$

** Notice, The disk or washer always comes from a slice perpendicular to The axis of revolution.