Math 20200
Calculus II

Lesson 17
Areas Between Curves

Dr. A. Marchese, The City College of New York

Table of Contents:

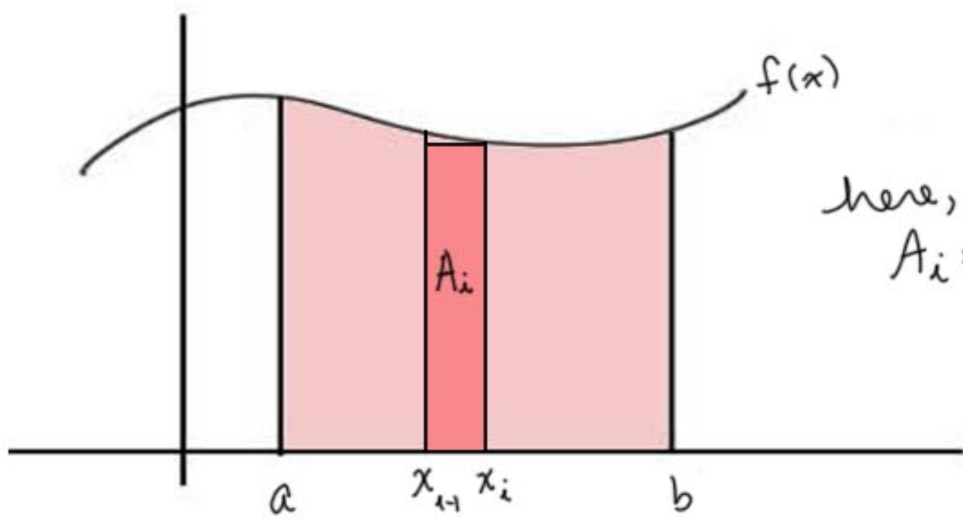
Areas Between Curves

In Calc I we learn that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i)\,\Delta x}_{\text{area of rectangle on subinterval}} \qquad n = \#\text{ subintervals}$$

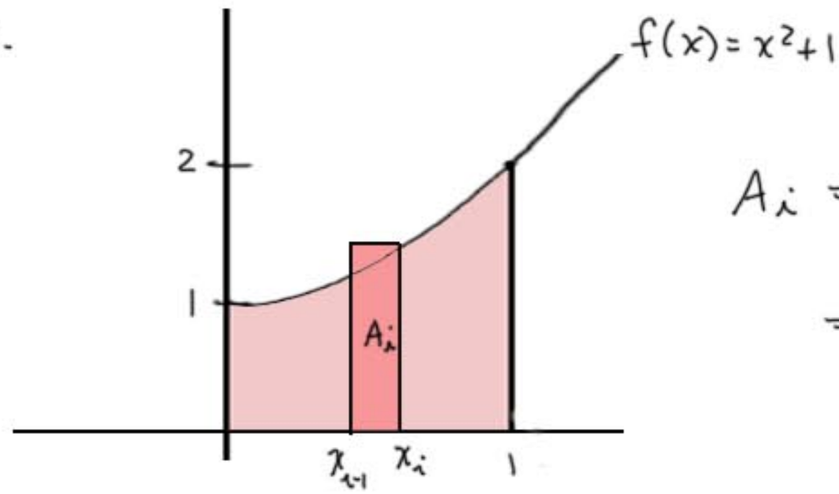
$$= \lim_{n\to\infty} \sum_{i=1}^{n} A_i \qquad A_i = \text{area over the } i^{\text{th}} \text{ subinterval}$$

$= \text{area under the graph of } f(x) \quad (\text{for } f(x) \geq 0)$



here,
$$A_i = \left(f(x_i) - 0\right)\Delta x$$

Ex.



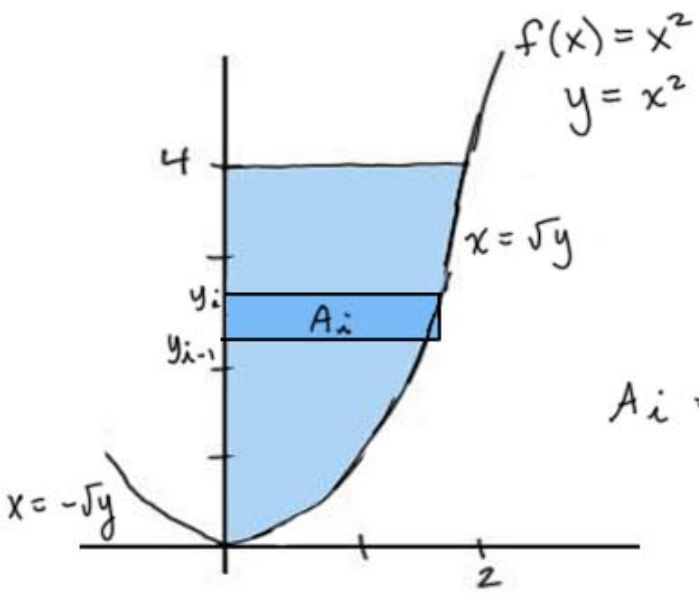
$$A_i = \left(f(x_i) - 0\right)\Delta x$$

$$= \left(x_i^2 + 1\right)\Delta x$$

and total area under curve $= \displaystyle\int_0^1 (x^2+1)\,dx$

$$= \left[\frac{x^3}{3} + x\right]_0^1 = \left(\frac{1}{3}+1\right) - \left(\frac{0^3}{3}+0\right) = \frac{4}{3}.$$

Now consider



$\Rightarrow x = \pm\sqrt{y}$

here we want horizontal rectangles

$$A_i = \left(\sqrt{y_i} - 0\right)\Delta y$$

and total area $= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\sqrt{y_i}\,\Delta y}_{A_i}$

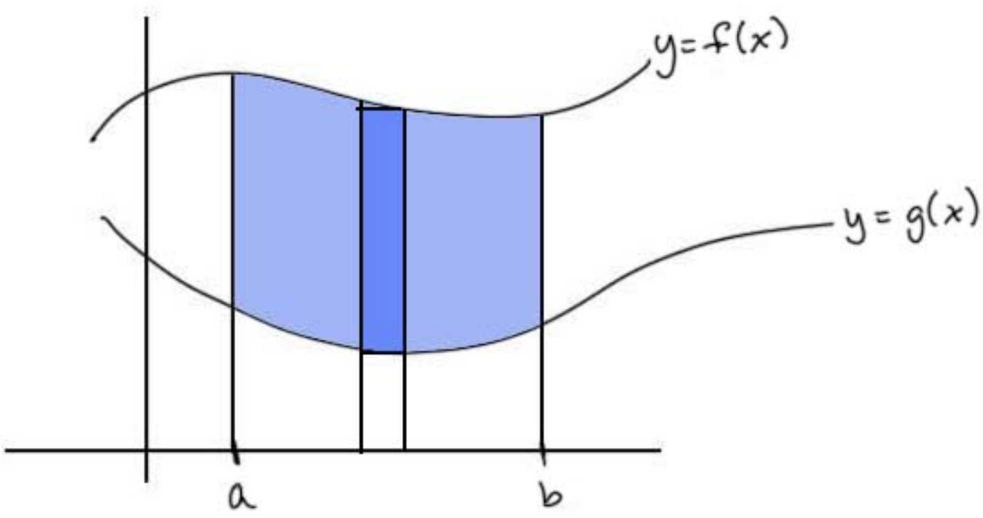
$$= \int_0^4 \sqrt{y}\, dy$$

$$= \int_0^4 y^{1/2}\, dy = \frac{2}{3} y^{3/2} \Big|_0^4 = \left(\frac{2}{3}\cdot 4^{3/2}\right) - \left(\frac{2}{3}\cdot 0^{3/2}\right)$$

$$= \frac{2}{3}\cdot 2^3 = \frac{16}{3}.$$

Area Between Curves:

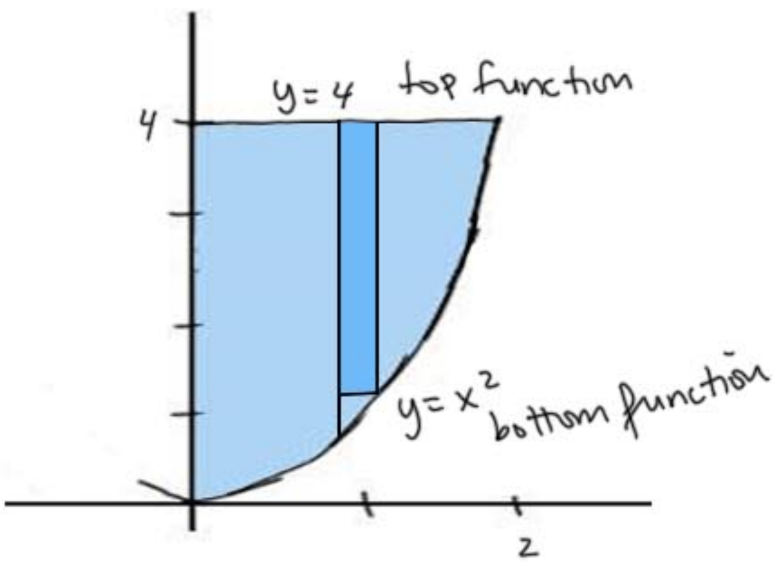
$y = f(x)$, $y = g(x)$ for $x \in [a, b]$



$$\text{area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \left(f(x) - g(x)\right) dx$$

$$\text{always } \int_a^b (\text{top} - \text{bottom})\,dx.$$

then notice the area we had above



can be computed as

$$\int_0^2 (4 - x^2)\,dx$$

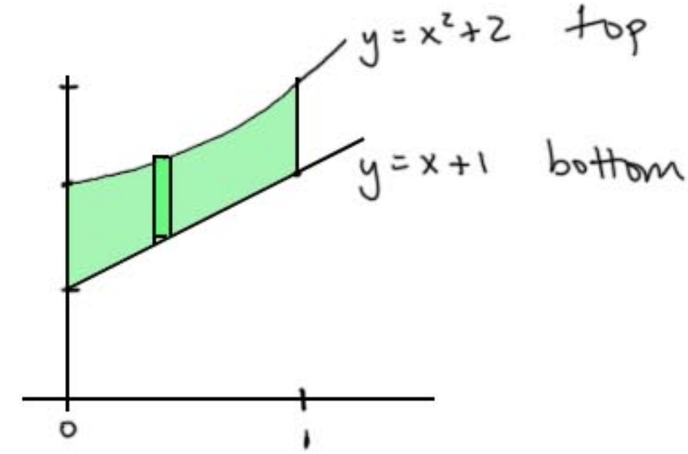
top   bottom   both functions of x

$$\int_0^2 (4-x^2)\,dx = 4x - \frac{x^3}{3}\Big|_0^2 = \left(4(2) - \frac{2^3}{3}\right) - \left(4(0) - \frac{0^3}{3}\right)$$

$$= 8 - \frac{8}{3} = \frac{24}{3} - \frac{8}{3} = \frac{16}{3}$$ same answer.

Ex. Find the area bounded by $y = x^2 + 2$ and $y = x + 1$ for $0 \le x \le 1$.

Start by sketching:



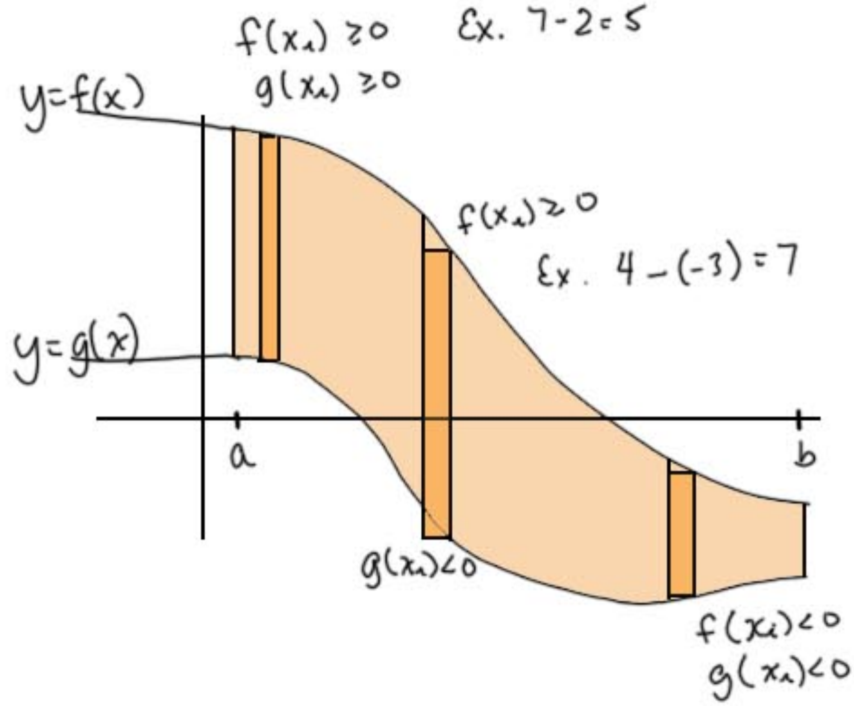
$$\text{area} = \int_0^1 \left((x^2+2) - (x+1) \right) dx = \int_0^1 (x^2 - x + 1)\, dx$$

$+2 \; -1$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + x \right]_0^1 = \left(\frac{1}{3} - \frac{1}{2} + 1 \right) - (0)$$

$$= \frac{2}{6} - \frac{3}{6} + \frac{6}{6} = \frac{5}{6}.$$

The functions above are all ≥ 0. What if one or both of the functions is < 0 over the interval?



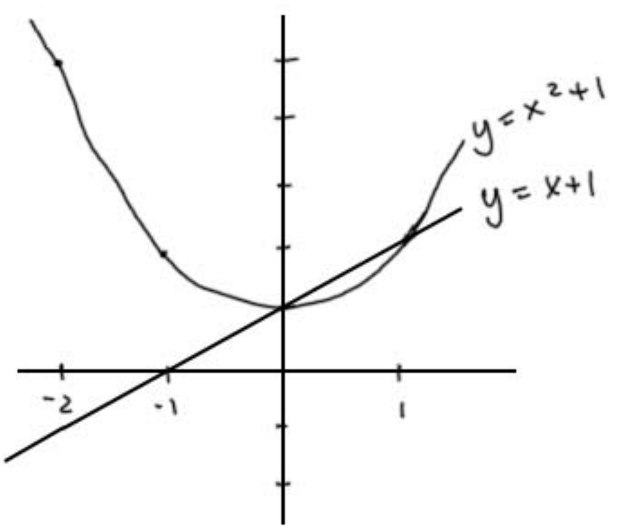
always
top - bottom.

Ex. $-2-(-5)$
$= -2+5 = 3$

Ex. Find The area bounded by $y = x^2 + 1$ and

$y = x + 1$ for $-2 \le x \le 1$

Start by sketching:



be sure to get correct points of intersection:

$x^2 + 1 = x + 1$

$x^2 = x$

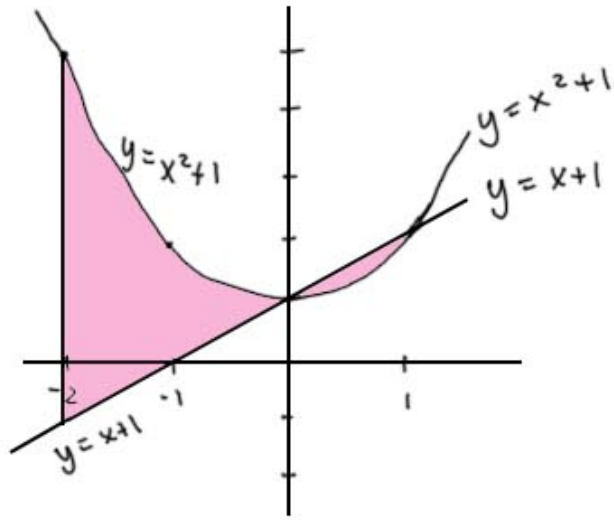
$x^2 - x = 0$

$x(x-1) = 0$

$x = 0,\ x - 1 = 0$

$x = 0,$
$x = 1.$

then:



notice that the "top" function changes over $-2 \le x \le 1$.

$\therefore$ we need to split the integral:

$$\text{shaded area} = \int_{-2}^{0} \Big(\underbrace{(x^2+1)-(x+1)}_{\text{simplify}}\Big)\,dx + \int_{0}^{1} \Big(\underbrace{(x+1)-(x^2+1)}_{\text{simplify}}\Big)\,dx$$

$$= \int_{-2}^{0} (x^2-x)\,dx + \int_{0}^{1} (x-x^2)\,dx =$$

$$= \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{0}^{1} =$$

$$= \left[\left(\frac{0^3}{3}-\frac{0^2}{2}\right)-\left(\frac{(-2)^3}{3}-\frac{(-2)^2}{2}\right)\right] + \left[\left(\frac{1^2}{2}-\frac{1^3}{3}\right)-\left(\frac{0^2}{2}-\frac{0^3}{3}\right)\right]$$

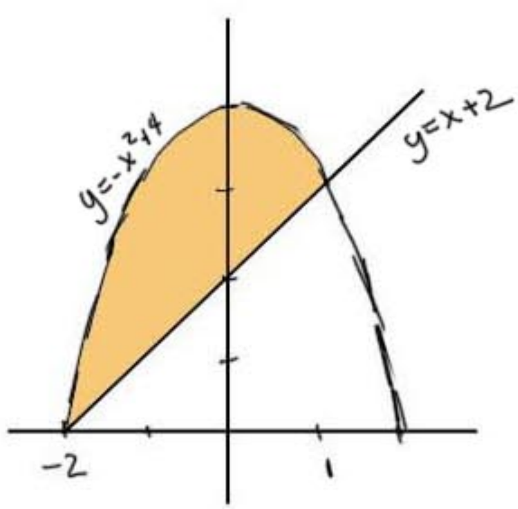
$= \quad -\left(\frac{-8}{3} - 2\right) + \left(\frac{1}{2} - \frac{1}{3}\right) =$

$= \quad \frac{16}{6} + \frac{12}{6} + \frac{3}{6} - \frac{2}{6} = \frac{29}{6}.$

Ex. Find the area bounded by $y = -x^2 + 4$ and
$y = x + 2$.

Start by sketching, be sure to find points of intersection

Work on this problem
on your own



points of intersection:

$-x^2 + 4 = x + 2$

$-x^2 - x + 2 = 0$

$x^2 + x - 2 = 0 \qquad x = -2$

$(x+2)(x-1) = 0 \qquad x = 1.$

$x + 2 = 0 \quad x - 1 = 0$

$$\int_{-2}^{1}\left((-x^2+4)-(x+2)\right)dx = \int_{-2}^{1}\left(-x^2-x+2\right)dx$$

$$= \left[-\frac{x^3}{3}-\frac{x^2}{2}+2x\right]_{-2}^{1}$$

$$-9/3 = -3$$

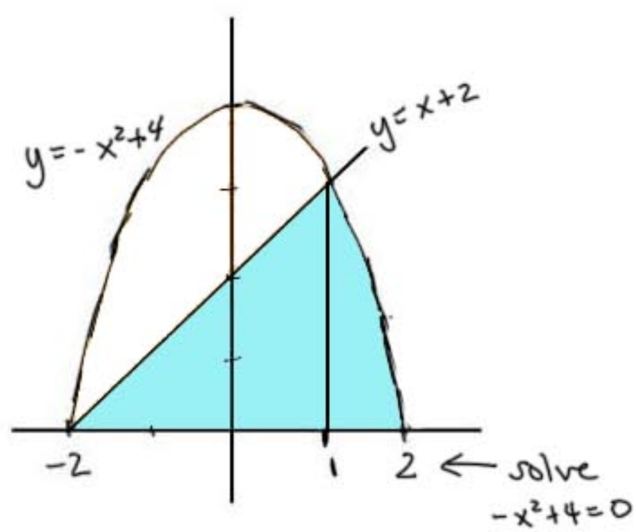
$$= \left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(\frac{8}{3}-\overset{2}{\cancel{\frac{4}{2}}}-4\right) =$$

$$-3+8-\frac{1}{2} = 5-\frac{1}{2} = \frac{9}{2}.$$

Ex. Find The shaded area

(same curves as above,

but different region)



To recognize top and bottom curves, we realize that the top curve changes at $x=1$. (bottom curve is $y=0$ for both)

$$\text{So total area} = \int_{-2}^{1}(x+2)\,dx + \int_{1}^{2}(-x^2+4)\,dx$$

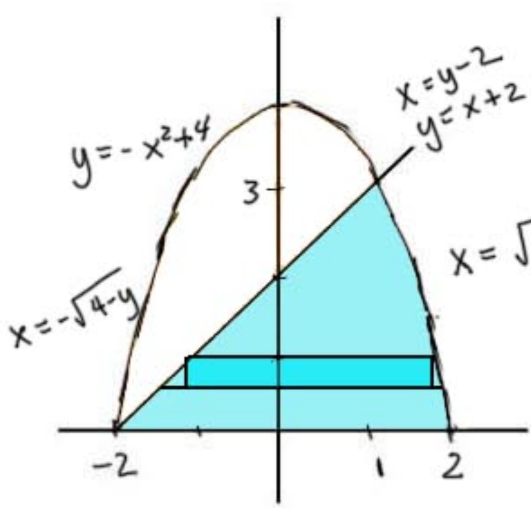
$$= \frac{x^2}{2}+2x\Big|_{-2}^{1} + \frac{-x^3}{3}+4x\Big|_{1}^{2} =$$

$$= \left(\tfrac{1}{2}+2\right)-\left(\tfrac{4}{2}-4\right) + \left(\tfrac{-8}{3}+8\right)-\left(\tfrac{-1}{3}+4\right) =$$

$$\tfrac{5}{2}+2 \qquad + \quad \tfrac{-7}{3}+4$$

$$= 6+\tfrac{15}{6}-\tfrac{14}{6} = 6\tfrac{1}{6} = \tfrac{37}{6}.$$

OR we could take horizontal rectangles:



and integrate

$$\int_{c}^{d}(\text{right} - \text{left})\,dy$$

$$\int_c^d (\text{right} - \text{left})\,dy = \int_0^3 \left(\sqrt{4-y} - (y-2)\right) dy$$

$$\uparrow$$

u-sub

$$= -\int_0^3 -\sqrt{4-y}\,dy - \int_0^3 (y-2)\,dy =$$

$u = 4-y \quad y=0,\ u=4$

$du = -dy \quad y=3,\ u=1$

$$= -\int_4^1 u^{1/2}\,du - \int_0^3 (y-2)\,dy =$$

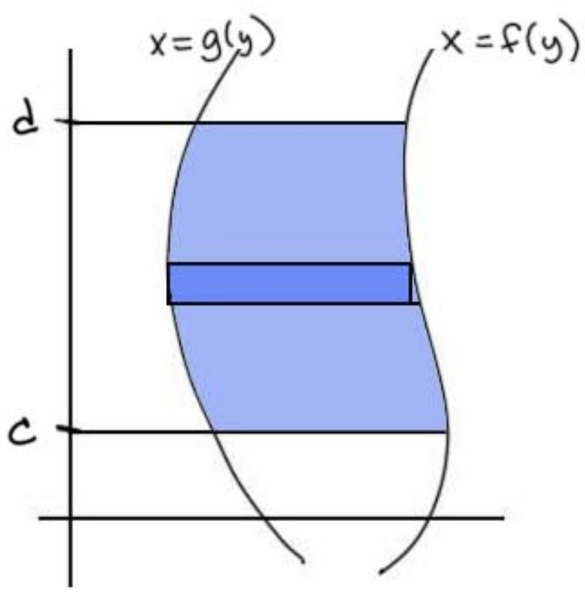
$$= \left[-\frac{2}{3}u^{3/2}\right]_4^1 - \left[\frac{y^2}{2} - 2y\right]_0^3$$

$$= \left[\left(-\frac{2}{3}\right) - \left(-\frac{2}{3}\cdot 8\right)\right] - \left[\left(\frac{9}{2} - 6\right) - 0\right]$$

$$= -\frac{2}{3} + \frac{16}{3} - \left(-\frac{3}{2}\right)$$

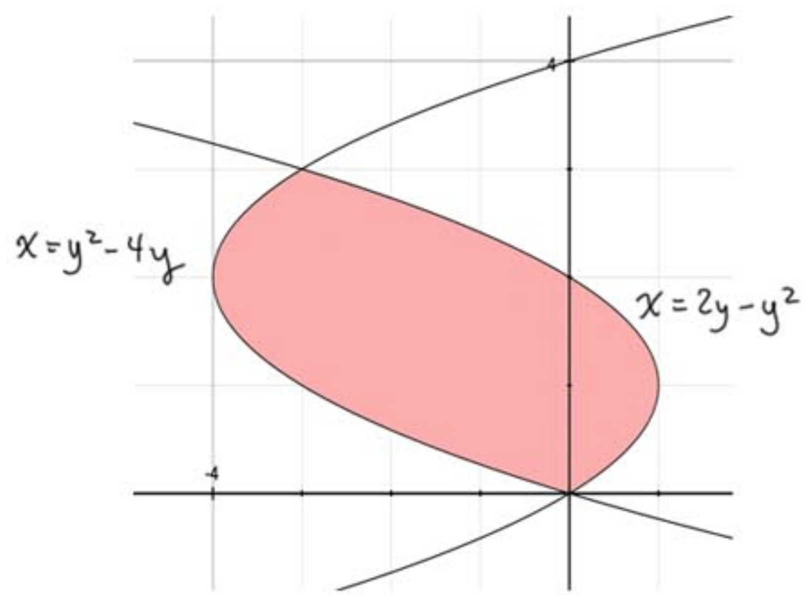
$$= \frac{14}{3} + \frac{3}{2} = \frac{28}{6} + \frac{9}{6} = \frac{37}{6}$$ same answer.

In general,



$$\text{area} = \int_c^d (\text{right} - \text{left})\,dy = \int_c^d \big(f(y) - g(y)\big)\,dy$$

Ex. Find the shaded area.



Work on this problem on your own

Find the points of intersection:

$$y^2 - 4y = 2y - y^2$$

$$2y^2 - 6y = 0$$

$$2y(y-3) = 0$$

$$2y = 0 \quad y - 3 = 0$$

$y = 0 \qquad x = y^2 - 4y = 0$

$y = 3 \qquad x = 3^2 - 4(3) = -3$

$(0,0)$
$(-3,3)$ points of intersection

$$\text{area} = \int_c^d (\text{right} - \text{left})\, dy =$$

$$= \int_0^3 \Big(\underbrace{(2y - y^2) - (y^2 - 4y)}_{\text{simplify}}\Big)\, dy$$

make sure you use parentheses!!!

$$= \int_0^3 (-2y^2 + 6y)\, dy = \left[-2\frac{y^3}{3} + \frac{6y^2}{2}\right]_0^3$$

$$= \left[-\frac{2}{3}y^3 + 3y^2\right]_0^3 = \left(-\frac{2}{3}(27) + 3(9)\right) - (0 + 0) =$$

$$= -18 + 27 = 9.$$

Summary: Finding Areas Between Curves:

1) sketch the region (this includes finding all points of intersection)

2) determine if the region has top and bottom curves (dx integral) or right and left curves (dy integral). **If the region can be described both ways, see which way is easier to set up/compute.