Math 20200
Calculus II

Lesson 16
The Comparison Theorem for Improper Integrals

Dr. A. Marchese, The City College of New York

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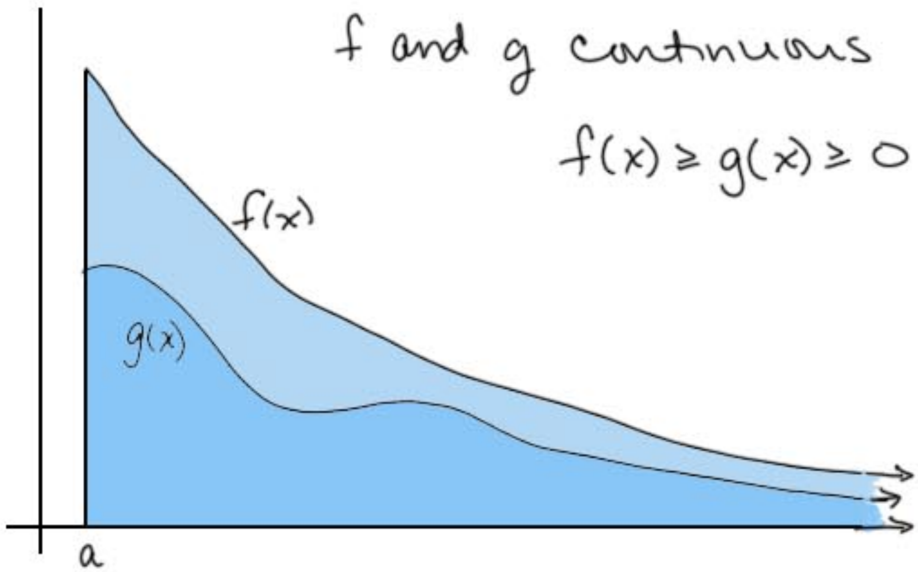
The Comparison Theorem for Improper Integrals

Suppose we have the following:



f and g continuous with

$f(x) \geq g(x) \geq 0 \quad \forall \ x \geq a$

If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

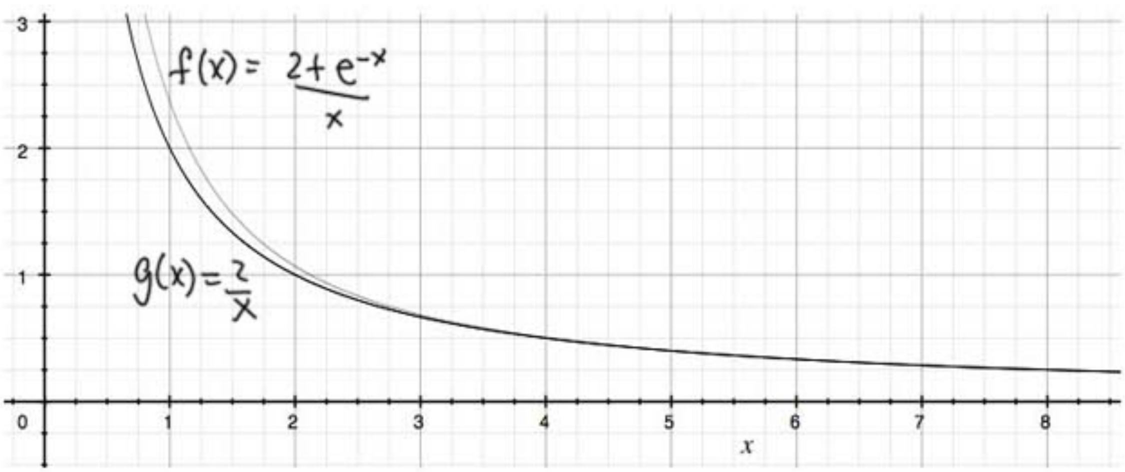
This is <u>The Comparison Theorem</u>.

Ex. $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$

We know $\displaystyle\frac{2+e^{-x}}{x} > \frac{2}{x}$ for $x \geq 1$

and $\displaystyle\int_1^\infty \frac{2}{x}\,dx = 2\int_1^\infty \frac{1}{x}\,dx$ diverges $\left(\frac{1}{x^p}\ p=1\right)$

$\therefore \displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ diverges



Recall:

⊛ $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$, diverges for $p \leq 1$

$\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges for $p<1$, diverges for $p \geq 1$

Ex. $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$

$$\frac{x}{\sqrt{1+x^6}} < \frac{x}{\sqrt{x^6}} \underset{\substack{\text{for}\\ x\geq 1}}{\overset{\downarrow}{=}} \frac{x}{x^3} = \frac{1}{x^2}$$

and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges $\left(\frac{1}{x^p}\ p=2\right)$

$\therefore \displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ converges also.

Ex. $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx \qquad \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ since $e^{-x}\leq 1$ on $[0,1]$

and $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges,

$\therefore \displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ also converges.

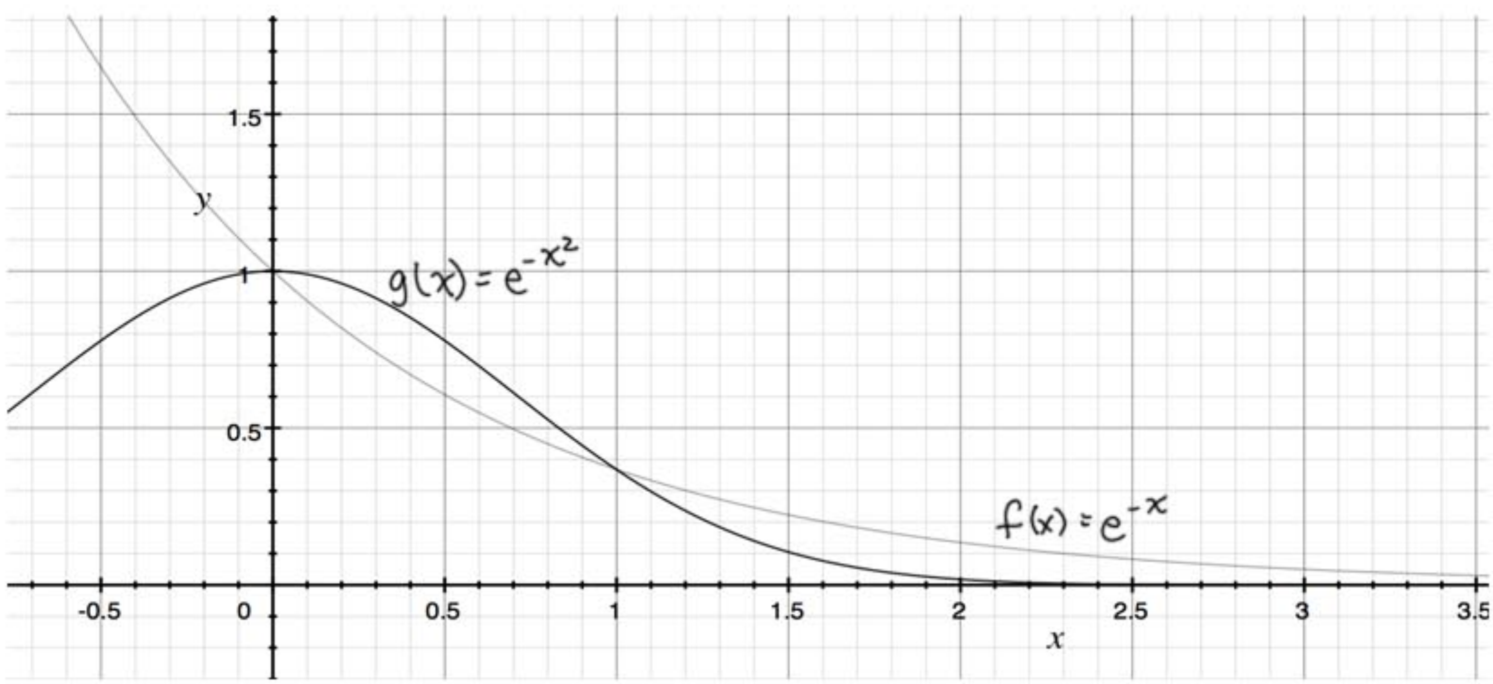
Remember that you can always check your $\frac{1}{x^p}$ integrals

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+} \int_a^1 x^{-1/2}\,dx = \lim_{a\to 0^+} 2x^{1/2}\Big]_a^1$$

$$= \lim_{a\to 0^+} \left(2\sqrt{1} - 2\sqrt{a}\right) = 2 \quad \text{converges.}$$

Ex. $\int_0^\infty e^{-x^2}dx$ does this converge or diverge?

Comparing e^{-x^2} to e^{-x} :



$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b =$$

$$= \lim_{b\to\infty} -\frac{1}{e^x}\Big|_1^b = \lim_{b\to\infty}\left(\cancel{-\frac{1}{e^b}} \to 0\right) - \left(\frac{-1}{e}\right) = 0 + \frac{1}{e} = \frac{1}{e}$$

converges.

So from $1 \to \infty$, we use The Comparison Theorem

and $\therefore \int_1^\infty e^{-x^2}\,dx$ converges.

notice $\int_0^1 e^{-x^2}\,dx$ is finite

and $\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$

$\therefore \int_0^\infty e^{-x^2}\,dx$ converges.