# Math 20200
# Calculus II

# Lesson 15
# Improper Integrals

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# Improper Integrals

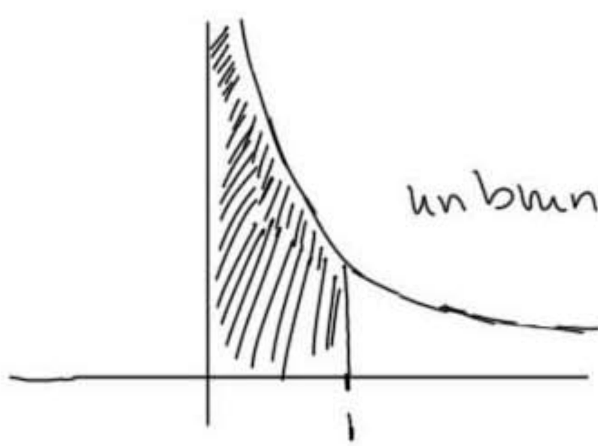
so far we've seen $\int_a^b f(x)\,dx$ where $a$ & $b$ are finite,

and where $f(x)$ exists on $[a,b]$.

but, what about: $\int_0^1 \frac{1}{x}\,dx$ ?

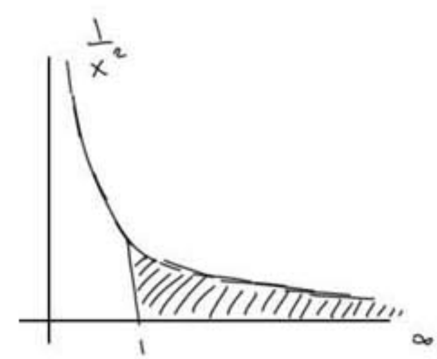
$0 \leftarrow$ x can't = 0 in $\frac{1}{x}$

$\frac{1}{0}$ = infinite behavior

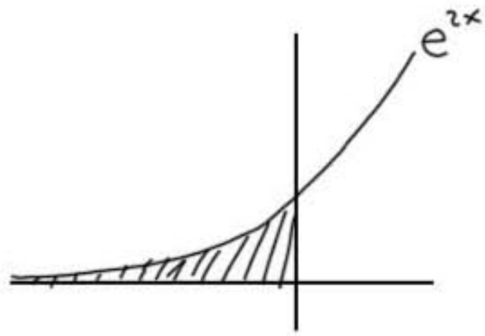


unbounded area, but might have finite value.

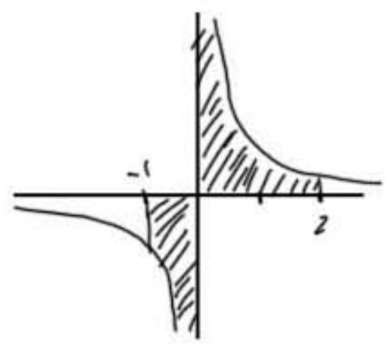
and what about $\int_1^\infty \frac{1}{x^2}\,dx$



or $\int_{-\infty}^{0} e^{2x}\,dx$




or $\int_{-1}^{2} \frac{1}{x}\,dx$




discontinuity at $x = 0$, in the interval over which we are integrating.

An integral is considered "improper" when:

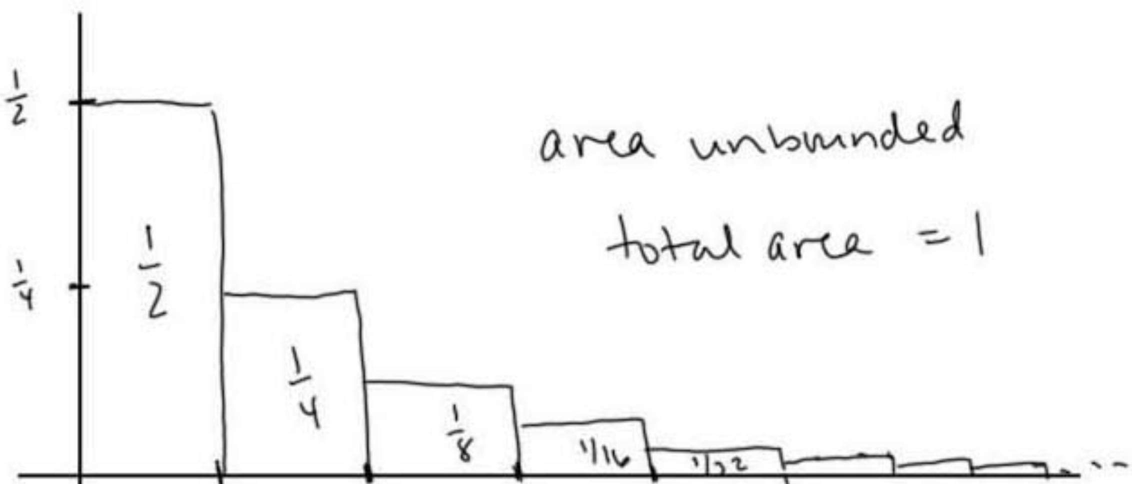
1) one or both of the bounds are infinite

2) there is a discontinuity over the interval $[a,b]$ over which we are integrating

How can an unbounded area have a finite measure?

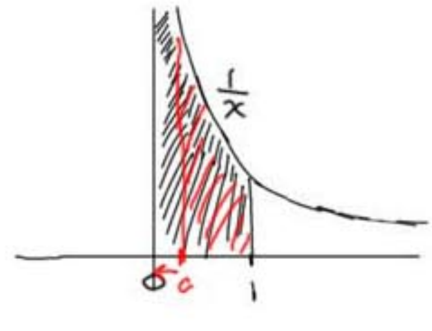
Consider:


so $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 1$



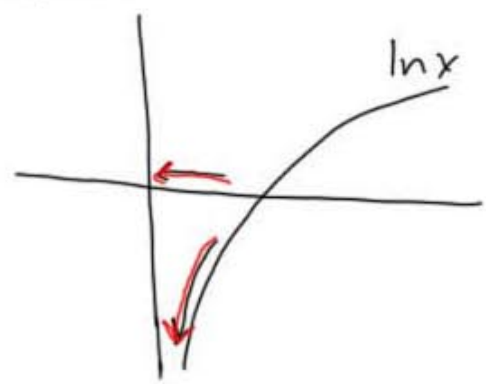
How to compute $\int_0^1 \frac{1}{x}\,dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{x}\,dx$

because x can't = 0,




$= \lim_{a \to 0^+} \ln|x| \Big]_a^1 = \lim_{a \to 0^+} \ln x \Big]_a^1 =$

since x-values > 0



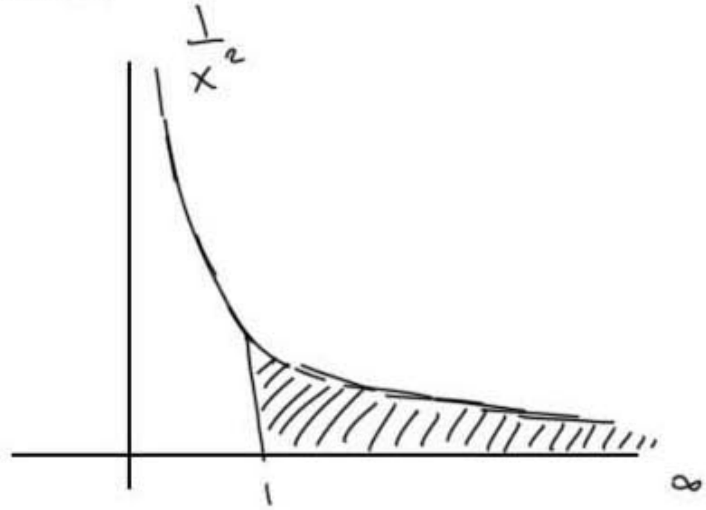

$= \lim_{a \to 0^+} \left( \underbrace{\ln 1}_{0} - \underbrace{\ln a}_{-\infty} \right)$

$= -(-\infty) = +\infty$

infinite area, and the integral diverges.

(if improper integral = finite #, then converges.
otherwise, diverges).

Ex. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ ← infinite bound.




$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} x^{-2}\,dx$$

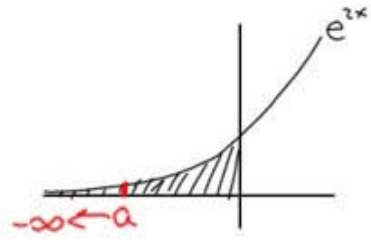
$$= \lim_{b\to\infty} \left.\frac{x^{-1}}{-1}\right]_1^b = \lim_{b\to\infty} \left.-\frac{1}{x}\right]_1^b =$$

$$= \lim_{b\to\infty} \left[\left(-\frac{1}{b}\right) - \left(-\frac{1}{1}\right)\right] = \lim_{b\to\infty} \left[-\cancelto{0}{\frac{1}{b}} + 1\right]$$

$$= 1$$

integral converges to 1. unbounded, finite area.

Ex. $\displaystyle\int_{-\infty}^{0} e^{2x}\,dx = \lim_{a\to-\infty} \int_{a}^{0} e^{2x}\,dx$




$$= \lim_{a\to-\infty} \left.\frac{1}{2}e^{2x}\right]_a^0 = \lim_{a\to-\infty}\left[\left(\frac{1}{2}e^{2(0)}\right) - \left(\frac{1}{2}e^{2a}\right)\right]$$

$$= \lim_{a\to-\infty}\left[\frac{1}{2} - \frac{1}{2}\cancel{e^{2a}}^{\,0}\right] = \frac{1}{2}.$$

integral converges to $\frac{1}{2}$.

unbounded, finite area.

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Ex. $\displaystyle\int_{0}^{\infty} \frac{1}{x^3}\,dx$ ← infinite limit

↖ $x=0$ not in domain of $\frac{1}{x^3}$

when both bounds are improper, split into two integrals

$$\int_{0}^{\infty} \frac{1}{x^3}\,dx = \int_{0}^{1} \frac{1}{x^3}\,dx + \int_{1}^{\infty} \frac{1}{x^3}\,dx$$

↙ split integral at any x-value in $(0, \infty)$

$$= \lim_{a\to 0^+} \int_a^1 \frac{1}{x^3}\,dx + \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx$$

$$= \lim_{a\to 0^+} \int_a^1 x^{-3}\,dx + \lim_{b\to\infty} \int_1^b x^{-3}\,dx$$

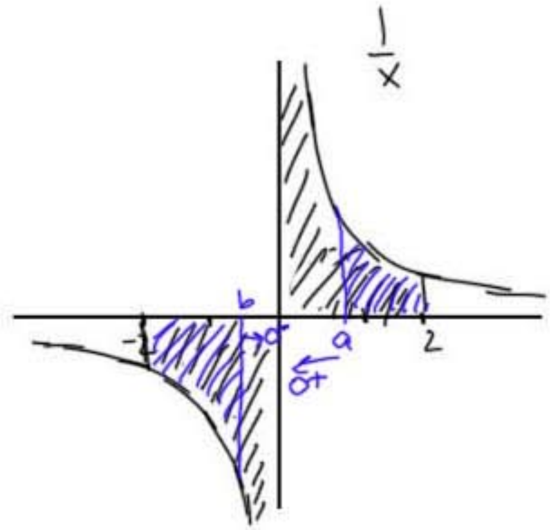
$$= \lim_{a\to 0^+} \left.\frac{x^{-2}}{-2}\right]_a^1 + \lim_{b\to\infty} \left.\frac{x^{-2}}{-2}\right]_1^b$$

$$= \lim_{a\to 0^+} \left.\frac{-1}{2x^2}\right]_a^1 + \lim_{b\to\infty} \left.\frac{-1}{2x^2}\right]_1^b$$

$$= \lim_{a\to 0^+} \underbrace{\left(\frac{-1}{2} + \overset{\infty}{+\frac{1}{2a^2}}\right)}_{-\frac{1}{2}+\infty=\infty} + \lim_{b\to\infty} \underbrace{\left(\overset{0}{\frac{-1}{2b^2}} - \frac{-1}{2}\right)}_{0+\frac{1}{2}=\frac{1}{2}} \overset{\infty+\frac{1}{2}=\infty}{=} \infty$$

integral diverges (as soon as we see part of the integral is infinite, integral diverges).

Consider $\int_{-2}^{2} \frac{1}{x}\,dx$




WRONG :

$$\int_{-2}^{2} \frac{1}{x}\,dx \cancel{=} \ln|x|\Big]_{-2}^{2} \cancel{= \ln 2 - \ln 2} \cancel{= 0}$$

$0 \in [-2,2]$ and $\frac{1}{0}$ DNE .

must separate:

$$\int_{-2}^{2} \frac{1}{x}\,dx = \int_{-2}^{0} \frac{1}{x}\,dx + \int_{0}^{2} \frac{1}{x}\,dx =$$

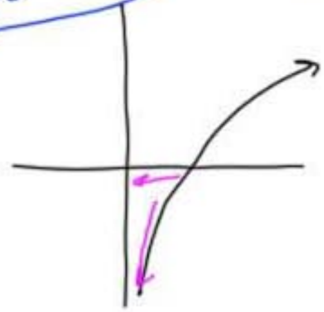
$$= \lim_{b \to 0^-} \int_{-2}^{b} \frac{1}{x}\,dx + \lim_{a \to 0^+} \int_{a}^{2} \frac{1}{x}\,dx$$

$$= \lim_{b\to 0^-} \ln|x|\Big]_{-2}^{b} + \lim_{a\to 0^+} \ln|x|\Big]_{a}^{2}$$

$$= \underbrace{\lim_{b\to 0^-}\Big[\underbrace{\ln|\underset{0^+}{b}|} - \ln|-2|\Big]}_{-\infty - \ln 2} + \underbrace{\lim_{a\to 0^+}\Big[\underbrace{\ln|2|}_{\ln 2} - \underbrace{\ln|a|}_{-\infty}\Big]}$$

$b \to 0^-$

$|b| \to 0^+$



$\int_{-2}^{0} \frac{1}{x}\,dx$ diverges

$\int_{0}^{2} \frac{1}{x}\,dx$ diverges.

$\Rightarrow \int_{-2}^{2} \frac{1}{x}\,dx$ diverges.

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(*) $\int_{1}^{\infty} \frac{1}{x^p}\,dx$ converges for $p>1$, diverges for $p\le 1$

$\int_{0}^{1} \frac{1}{x^p}\,dx$ converges for $p<1$, diverges for $p\ge 1$.

Ex. $\displaystyle\int_0^{\infty} x e^{-2x}\, dx = \lim_{b \to \infty} \int_0^{b} x e^{-2x}\, dx$

by parts


Work on this problem
on your own

$u = x \qquad du = dx$

$dv = e^{-2x}dx \qquad v = -\frac{1}{2}e^{-2x}$

$$\lim_{b \to \infty} \int_0^{b} x e^{-2x}\, dx =$$

$$= \lim_{b \to \infty} \left[ -\frac{1}{2}xe^{-2x} \Big|_0^b + \frac{1}{2}\int_0^b e^{-2x}\, dx \right]$$

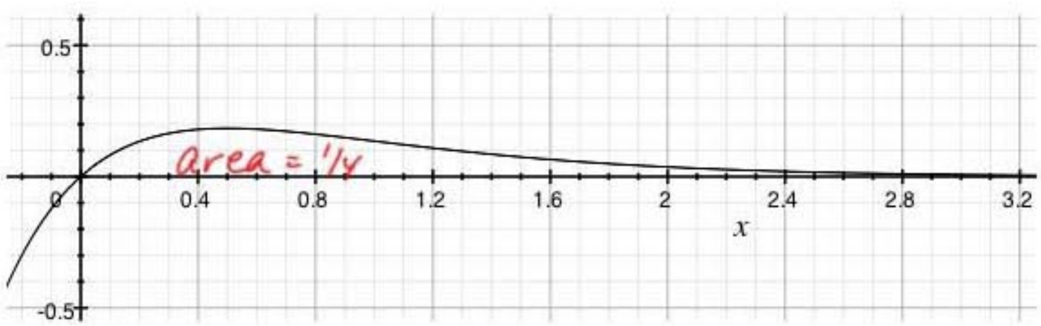
$$= \lim_{b\to\infty} \left[ -\frac{1}{2}xe^{-2x} - \frac{1}{4} e^{-2x}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\left(-\frac{1}{2} be^{-2b} - \frac{1}{4} e^{-2b}\right) - \left(0 - \frac{1}{4} e^{0}\right)\right]$$

$$= \lim_{b\to\infty}\left[ -\frac{1}{2} \underbrace{b}_{\infty}\underbrace{e^{-2b}}_{\cdot\ 0} - \cancelto{0}{\frac{1}{4} e^{-2b}} + \frac{1}{4}\right]$$

$$= \lim_{b\to\infty} -\frac{1}{2}\left(\frac{b}{e^{2b}}\right) + \frac{1}{4} \qquad \frac{\infty}{\infty} \text{ L'Hospital's}$$

$$\overset{LH}{=} \lim_{b\to\infty} -\frac{1}{2}\cancelto{0}{\left(\frac{1}{2e^{2b}}\right)} + \frac{1}{4} = \frac{1}{4}$$




integral converges to $\frac{1}{4}$

unbounded finite area

Ex. $\displaystyle\int_{-\infty}^{\infty} \cos \pi t \, dt$


Work on this problem
on your own

$$\int_{-\infty}^{\infty} \cos \pi t \, dt \underset{\substack{\uparrow \\ \text{split} \\ \text{any value} \\ \text{in } (-\infty,\infty)}}{=} \int_{-\infty}^{0} \cos \pi t \, dt + \int_{0}^{\infty} \cos \pi t \, dt$$

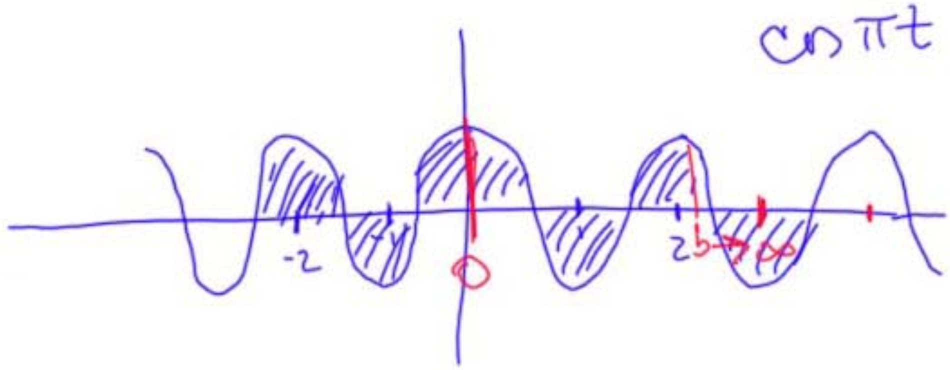
$$= \lim_{a \to -\infty} \int_{a}^{0} \cos \pi t \, dt + \lim_{b \to \infty} \int_{0}^{b} \cos \pi t \, dt$$

$$= \lim_{a \to -\infty} \frac{1}{\pi} \sin \pi t \Big]_{a}^{0} + \lim_{b \to \infty} \frac{1}{\pi} \sin \pi t \Big]_{0}^{b}$$

$$= \lim_{a\to-\infty}\left[\left(\underbrace{\tfrac{1}{\pi}\cancel{\sin\pi(0)}}_{0}\right)-\left(\tfrac{1}{\pi}\sin\pi a\right)\right]+\lim_{b\to\infty}\left[\left(\tfrac{1}{\pi}\sin\pi b\right)-(0)\right]$$

$$= \underbrace{\lim_{a\to-\infty} -\tfrac{1}{\pi}\sin\pi a}_{\substack{\text{DNE}\\ \text{oscillating}}} + \underbrace{\lim_{b\to\infty}\tfrac{1}{\pi}\sin\pi b}_{\substack{\text{DNE}\\ \text{oscillating}}}$$

$\therefore$ integral <u>diverges</u>.




Please view Lesson 16 for the comparison theorem for improper integrals.