Math 20200
Calculus II

Lesson 14
Approximate Integration

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Approximate Integration

Some functions are not the derivative of any others, so.... Can't find an antiderivative.

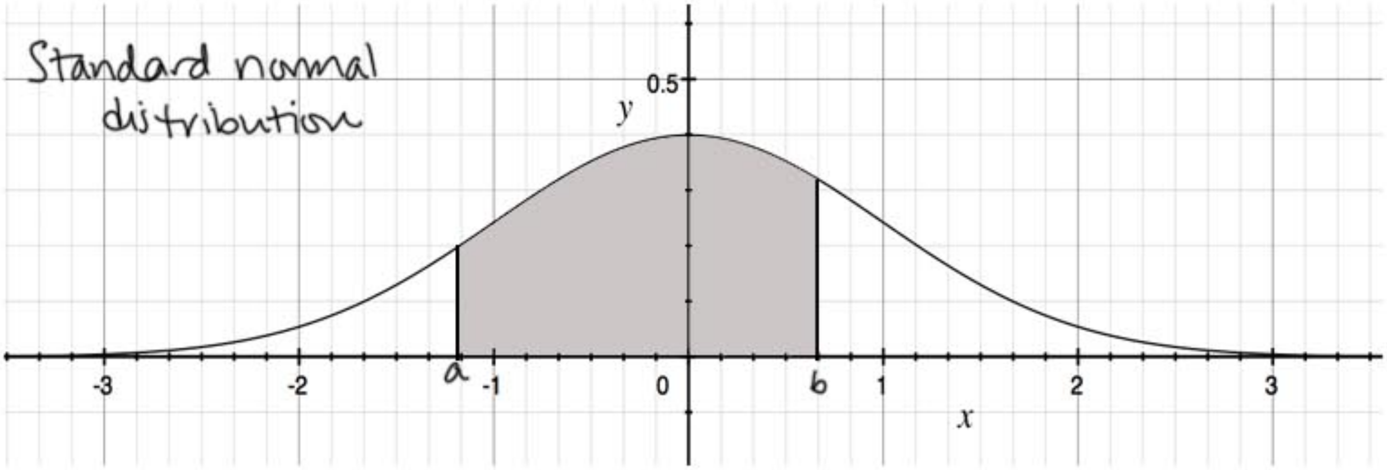
Ex. $\int e^{x^2}\,dx = ?$ no solution.

$\int e^{-\frac{x^2}{2}}\,dx = ?$ no solution

But, an important application:

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the standard normal curve

and $\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ = probability that the variable lies between $x = a$ + $x = b$.

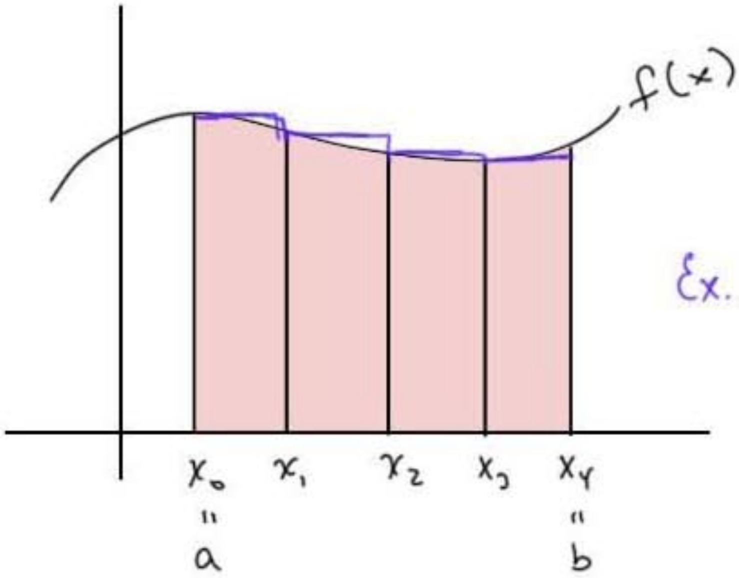


Although $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ does not have an antiderivative, we can approximate definite integrals.

In this lesson we'll approximate a simpler integral: $\int_0^2 e^{x^2}\,dx$.

First, recall Riemann Sums:

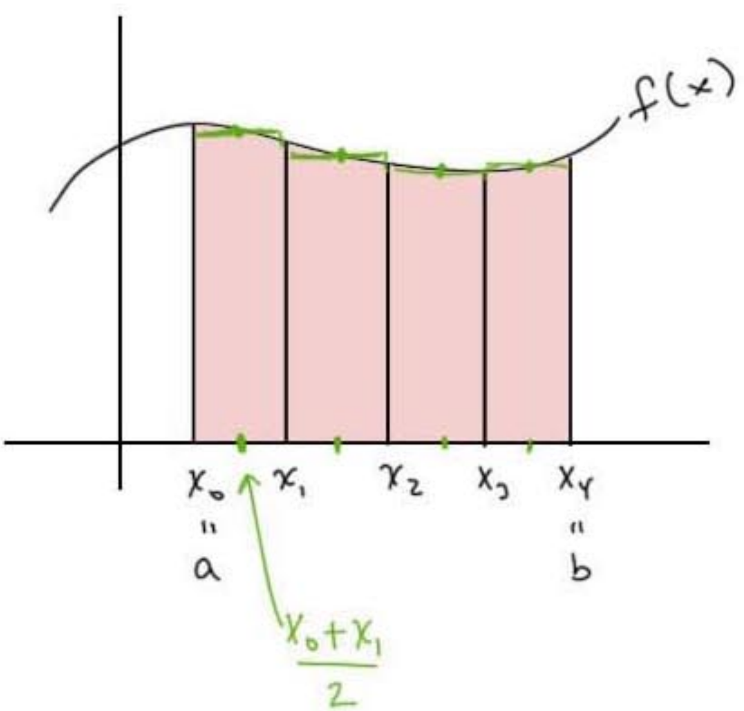
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i)\,\Delta x}_{\text{area of rectangle on subinterval}} \qquad n = \#\text{ subintervals}$$



$\Delta x = \frac{b-a}{n}$

Ex. left hand sum

Midpoint Rule uses the function value at the midpoint of each subinterval as the height of rectangle.



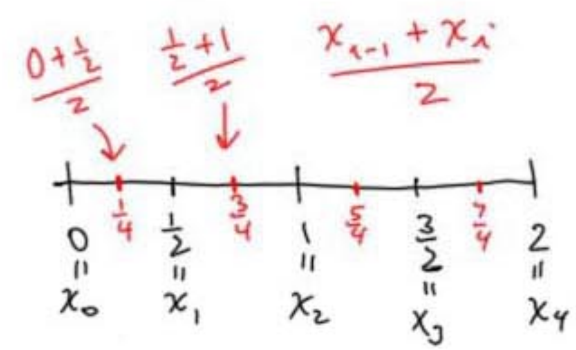
Midpoint Rule:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

Ex. $\int_0^2 e^{x^2}\,dx$ approximate using Midpoint Rule and $\underbrace{4}_{n}$ subintervals

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$



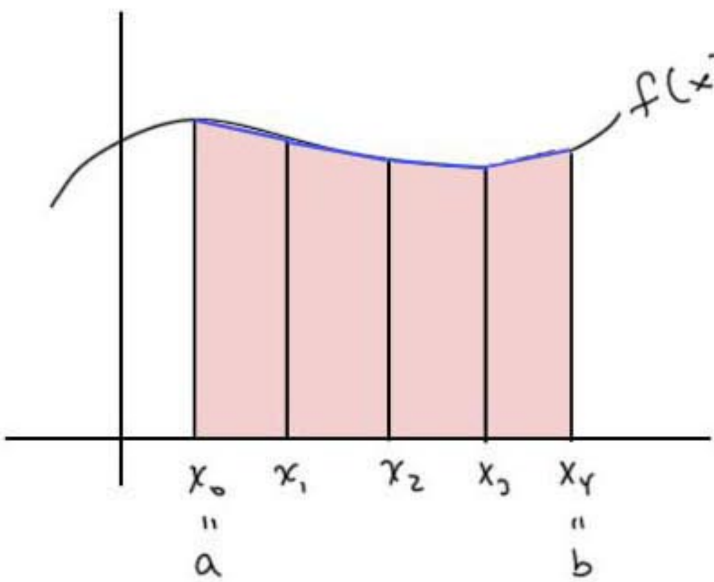
Midpt rule: $\int_0^2 e^{x^2}\,dx \approx \sum_{i=1}^{4} f\left(\frac{x_{i-1}+x_i}{2}\right)\overbrace{\Delta x}^{\frac{1}{2}}$

$$= \frac{1}{2}\left(f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right)\right)$$

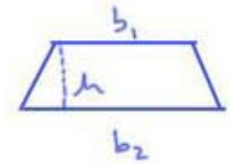
$$= \underbrace{\frac{1}{2}\left(e^{(1/4)^2} + e^{(3/4)^2} + e^{(5/4)^2} + e^{(7/4)^2}\right)}_{\text{leave here on exam.}} \approx 14.486$$

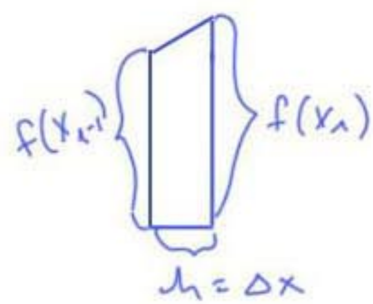
Trapezoid Rule

instead of using rectangles on each subinterval,
we use trapezoids



area trapezoid: $\frac{1}{2}h(b_1+b_2)$





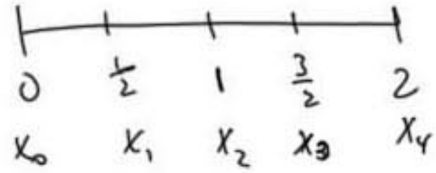
Trapezoid Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{1}{2}\Delta x\left(f(x_{i-1})+f(x_i)\right)$$

$$= \frac{1}{2}\Delta x\Big(f(x_0)+2f(x_1)+2f(x_2)+\cdots + 2f(x_{n-1})+f(x_n)\Big)$$

$$= \frac{b-a}{2n}\left(f(x_0)+2f(x_1)+2f(x_2)\cdots+2f(x_{n-1})+f(x_n)\right)$$

For all approximation methods:



$$\boxed{x_i = a + i\Delta x}$$

$$x_0 = a + 0(\Delta x) = a$$

$$x_1 = a + 1\Delta x$$
$$= 0 + 1\left(\tfrac{1}{2}\right) = \tfrac{1}{2}$$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$$x_2 = a + 2\Delta x$$
$$= 0 + 2\left(\tfrac{1}{2}\right) = 1$$

Trapezoid Rule.

$$\frac{b-a}{2n} = \frac{2-0}{2(4)} = \frac{2}{8} = \frac{1}{4}$$

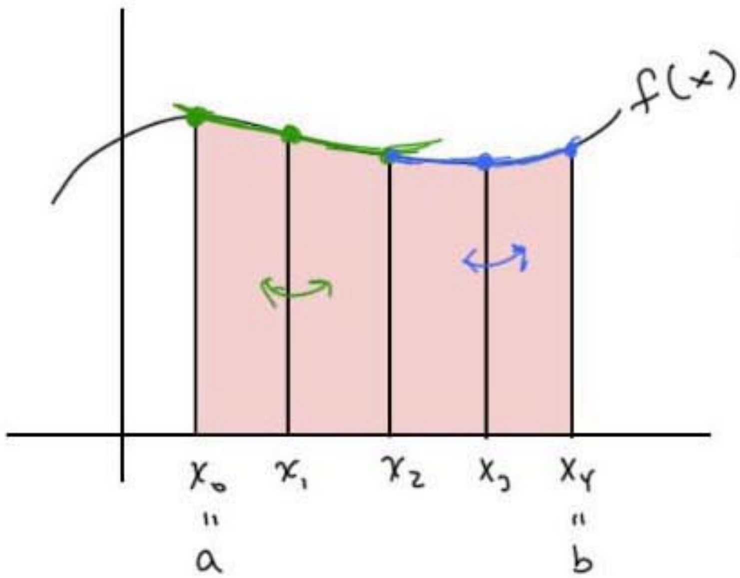
$$\int_0^2 e^{x^2}\,dx = \boxed{\tfrac{1}{4}}\left(f(\underbrace{0}_{x_0}) + 2f(\underbrace{\tfrac{1}{2}}_{x_1}) + 2f(\underbrace{1}_{x_2}) + 2f(\underbrace{\tfrac{3}{2}}_{x_3}) + f(\underbrace{2}_{x_4}) \right)$$

$$= \tfrac{1}{4}\left(e^{(0)^2} + 2e^{(1/2)^2} + 2e^{(1)^2} + 2e^{(3/2)^2} + e^{2^2} \right)$$

$$= \underbrace{\tfrac{1}{4}\left(1 + 2e^{1/4} + 2e + 2e^{9/4} + e^{4} \right)}_{\text{leave here on exam}} \approx 20.645$$

Simpson's Rule: uses on each 2 intervals,
a parabola fitting the 3pts.

area under parabola on those intervals



(n must be even)

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)\overbrace{+\cdots+}^{\text{alternating } 2+4}4f(x_{n-1})+f(x_n)\right]$$

side note:

how to fit a parabola through 3 pts:

(x_1,y_1) (x_2,y_2) (x_3,y_3)

parabola = $y = ax^2+bx+c$

$\therefore \quad y_1 = a(x_1)^2 + b\,x_1 + c$

$y_2 = a(x_2)^2 + b\,x_2 + c$

$y_3 = a(x_3)^2 + b\,x_3 + c$

3 equations in a b + c

solve for a, b, c.

Simpson's Rule: $\frac{b-a}{3n} = \frac{2-0}{3(4)} = \frac{2}{12} = \frac{1}{6}$

$$\int_0^2 e^{x^2}\,dx \approx \frac{1}{6}\Big(f(\underbrace{0}_{x_0}) + 4f(\underbrace{\tfrac{1}{2}}_{x_1}) + 2f(\underbrace{1}_{x_2}) + 4f(\underbrace{\tfrac{3}{2}}_{x_3}) + f(\underbrace{2}_{x_4})\Big)$$

$$= \frac{1}{6}\left(e^{(0)^2} + 4e^{(1/2)^2} + 2e^{1^2} + 4e^{(3/2)^2} + e^{2^2}\right)$$

$$= \underbrace{\frac{1}{6}\left(1 + 4e^{1/4} + 2e + 4e^{9/4} + e^{4}\right)}_{\text{leave here on exam}} \approx 17.354.$$

* the more subintervals we use, the better the approx.

Ex. Compute $\int_1^4 \frac{1}{x^2}\,dx$ exactly,

then approximate: $n=3$ for Midpoint Rule

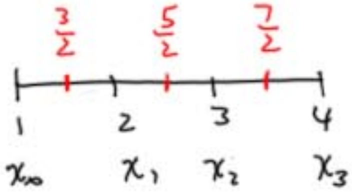
$n=6$ for Trapezoid and Simpson's Rules.

STOP Work on this problem on your own

$$\int_1^4 \frac{1}{x^2}\,dx = \int_1^4 x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right|_1^4 = \left.\frac{-1}{x}\right|_1^4 =$$

$$= \left(-\frac{1}{4}\right) - \left(-\frac{1}{1}\right) = -\frac{1}{4} + 1 = \boxed{\frac{3}{4}} \text{ exact answer.}$$

Midpoint Rule:
$n = 3$



$x_i = a + i\Delta x$

$\Delta x = \frac{b-a}{n} = \frac{4-1}{3} = \frac{3}{3} = 1$

$x_i = 1 + i(1)$
$= 1 + i.$

evaluation points are the midpts of the intervals

$$\int_1^4 \frac{1}{x^2}\,dx \approx \Delta x\left(f\left(\tfrac{3}{2}\right) + f\left(\tfrac{5}{2}\right) + f\left(\tfrac{7}{2}\right)\right) = 1\left(\frac{1}{\left(\frac{3}{2}\right)^2} + \frac{1}{\left(\frac{5}{2}\right)^2} + \frac{1}{\left(\frac{7}{2}\right)^2}\right)$$

$$= \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{5}\right)^2 + \left(\tfrac{2}{7}\right)^2 = \frac{4}{9} + \frac{4}{25} + \frac{4}{49} = \frac{4900 + 1764 + 900}{11025}$$

$$= \frac{7564}{11025} \approx .686$$

exact $= .75$

Trapezoid Rule:

$n = 6$

$\Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$

1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
x_0	x_1	x_2	x_3	x_4	x_5	x_6

$$\int_1^4 \frac{1}{x^2}\,dx \approx \frac{1}{2}\Delta x\left(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+f(x_6)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{1^2}+2\cdot\frac{1}{\left(\frac{3}{2}\right)^2}+2\cdot\frac{1}{2^2}+2\cdot\frac{1}{\left(\frac{5}{2}\right)^2}+2\cdot\frac{1}{3^2}+2\cdot\frac{1}{\left(\frac{7}{2}\right)^2}+\frac{1}{4^2}\right)$$

$$= \frac{1}{4}\left(1+2\cdot\frac{4}{9}+\frac{1}{2}+2\cdot\frac{4}{25}+\frac{2}{9}+2\cdot\frac{4}{49}+\frac{1}{16}\right)$$

$$= \frac{1}{4}\left(1+\frac{8}{9}+\frac{1}{2}+\frac{8}{25}+\frac{2}{9}+\frac{8}{49}+\frac{1}{16}\right)$$

$9\cdot 25\cdot 49\cdot 16 = 176{,}400$

$$= \frac{1}{4}\left(\frac{176{,}400+156{,}800+88{,}200+56{,}448+39{,}200+28800+11025}{176{,}400}\right)$$

$$= \frac{1}{4}\left(\frac{556873}{176{,}400}\right) = \frac{556873}{705600} \approx .789 \qquad \text{exact} = .75.$$

Simpson's Rule:

$n=6$

$\Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$

1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
x_0	x_1	x_2	x_3	x_4	x_5	x_6

$$\int_1^4 \frac{1}{x^2}\,dx \approx \frac{1}{3}\Delta x\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+f(x_6)\right)$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{1}{1^2}+4\cdot\frac{1}{\left(\frac{3}{2}\right)^2}+2\cdot\frac{1}{2^2}+4\cdot\frac{1}{\left(\frac{5}{2}\right)^2}+2\cdot\frac{1}{3^2}+4\cdot\frac{1}{\left(\frac{7}{2}\right)^2}+\frac{1}{4^2}\right)$$

$$= \frac{1}{6}\left(1+4\cdot\frac{4}{9}+\frac{1}{2}+4\cdot\frac{4}{25}+\frac{2}{9}+4\cdot\frac{4}{49}+\frac{1}{16}\right)$$

$$= \frac{1}{6}\left(\frac{88769}{19600}\right) = \frac{88769}{117600} \approx .755 \qquad \text{exact} = .75$$

Error Bounds on Numerical Integration

Suppose f'' is continuous on $[a,b]$.

and $|f''(x)| \le K \quad \forall x \in [a,b]$

then $|ET_n| \le \dfrac{K(b-a)^3}{12n^2}$ — Error using Trapezoid Rule, n subdivisions

$|EM_n| \le \dfrac{K(b-a)^3}{24n^2}$ — Error using Midpoint Rule, n subdivisions

Suppose $f^{(iv)}$ continuous on $[a,b]$ +

$$|f^{(iv)}(x)| \leq L \quad \forall x \in [a,b]$$

then $$|ES_n| \leq \frac{L(b-a)^5}{180n^4}$$

Error using Simpson's Rule

n subdivisions

Ex. find the number of subintervals n so that the approximation to $\int_0^2 e^{x^2}\,dx$ using the Trapezoid Rule has error $\leq 10^{-3}$.

$$|ET_n| \leq \frac{K(b-a)^3}{12n^2} \overset{\text{need}}{\leq} 10^{-3}$$

need K, $|f''(x)| \leq K$ on $[0,2]$.

$f(x) = e^{x^2} \quad f'(x) = 2xe^{x^2} \quad f''(x) = 2e^{x^2} + 2x\,2xe^{x^2}$

$$= \underbrace{\underbrace{2e^{x^2}}_{>0} \underbrace{(1+2x^2)}_{>0}}_{>0}$$

$|f''(x)| = f''(x)$ since f'' always positive

need abs max of $f''(x)$ on $[0,2]$

need $f'''(x)$

$$f'''(x) = 2 \cdot 2x e^{x^2}(1+2x^2) + 2e^{x^2}(4x)$$

$$= 4xe^{x^2}(1+2x^2+2) = \underbrace{4x}_{\geq 0} \underbrace{e^{x^2}}_{>0} \underbrace{(3+2x^2)}_{>0}$$

$x \in [0,2]$

$\therefore$ f'' increasing on $[0,2]$

abs max of f'' on $[0,2]$ occurs at $x=2$.

$$f''(x) = 2e^{x^2}(1+2x^2) \qquad f''(2) = 2e^4(1+2(4))$$

$$= 18e^4 = K.$$

$$|ET_n| \leq \frac{18e^4(2)^3}{12n^2} \leq 10^{-3}$$

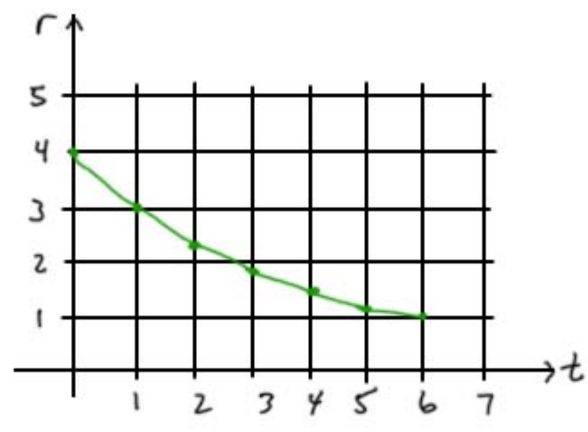
$$\frac{12e^4}{n^2} \le 10^{-3}$$

$$12000e^4 \le n^2$$

$$n \ge 809.43$$

$$\boxed{n \ge 810}$$

Ex. Oil leaked from a tank at a rate of $r(t)$ liters per hour, where the graph of r is shown below. Use Simpson's Rule to estimate the total amount of oil that leaked out during the first six hours.



We need to estimate $\int_0^6 r(t)\,dt$

Recall: $\int_a^b (\text{rate of change function})\,dt = $ total change over $[a,b]$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)\overbrace{+\cdots}^{\text{alternating } 2 \,\&\, 4}+4f(x_{n-1})+f(x_n)\right]$$

$a=0,\ b=6$ let's use 6 subintervals $(n=6)$

then $\Delta t = \frac{b-a}{n} = \frac{6-0}{6} = 1$, so
$x_0 = a = 0$
$x_1 = a + 1\Delta t = 1$
$x_2 = a + 2\Delta t = 2$
$x_3 = 3 \quad x_4 = 4 \quad x_5 = 5 \quad x_6 = 6$

$$\int_0^6 r(t)\,dt \approx \frac{6-0}{3(6)}\left[\underbrace{r(0)}_{4}+4\underbrace{r(1)}_{3}+2\underbrace{r(2)}_{\approx 2.4}+4\underbrace{r(3)}_{\approx 1.8}+2\underbrace{r(4)}_{\approx 1.5}+4\underbrace{r(5)}_{\approx 1.2}+\underbrace{r(6)}_{1}\right]$$

$$\approx \frac{1}{3}\left[4+4(3)+2(2.4)+4(1.8)+2(1.5)+4(1.2)+1\right]$$

$$= \frac{1}{3}\left[4+12+4.8+7.2+3+4.8+1\right]$$

$$= \frac{1}{3}\left[36.8\right] \approx 12.27 \text{ liters.}$$