

# Math 20200

## Calculus II

### Lesson 13

## Partial Fractions

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# Partial Fractions

a method for integrating a proper rational function. ← degree of numerator is less than degree of denominator

The idea of the method of partial fractions comes from making a common denominator to add/subtract fractions:

$$\frac{2}{x+2} + \frac{3}{x-1} = \frac{2(x-1) + 3(x+2)}{(x+2)(x-1)} = \frac{5x+4}{x^2+x-2}$$

then  $\int \frac{5x+4}{x^2+x-2} dx = \int \left( \frac{2}{x+2} + \frac{3}{x-1} \right) dx =$

$$= \int \frac{2}{x+2} dx + \int \frac{3}{x-1} dx =$$

$u = x+2$   
 $du = dx$

$$= \int \frac{2}{u} du$$

$$= 2 \ln|u|$$

$$= 2 \ln|x+2| + 3 \ln|x-1| + C$$

When  $u = x + k$   
↑  
coefficient = 1  
 $du = dx$   
↓

Steps to using The method of partial fractions to integrate a proper rational function:

1) factor the denominator completely

2) set up partial fractions (with proper numerators)

- linear factors get constant numerators

- quadratic factors get linear numerators

- for repeated factors, "count up" to the multiplicity

3) find the coefficients (method ① or method ② below)

4) integrate!



method (2) for finding A + B (coefficients)

$$5x + 4 = A(x-1) + B(x+2) \quad \forall x$$

choose convenient values for x

$$x=1: 5(1)+4 = A(\cancel{0}) + B(3)$$

$$9 = 3B \Rightarrow \boxed{B=3}$$

$$x=-2: 5(-2)+4 = A(-3) + B(\cancel{0})$$

$$-6 = -3A \Rightarrow \boxed{A=2}$$

$$\therefore \frac{5x+4}{x^2+x-2} = \frac{2}{x+2} + \frac{3}{x-1}$$

$$\text{and } \int \frac{5x+4}{x^2+x-2} dx = \int \frac{2}{x+2} dx + \int \frac{3}{x-1} dx =$$

$$= 2 \ln|x+2| + 3 \ln|x-1| + C.$$

$$\text{Ex. } \int \frac{2x}{x^2-6x+9} dx \quad \text{Repeated Linear Factor}$$

$$\frac{2x}{x^2-6x+9} = \frac{2x}{(x-3)(x-3)} = \frac{2x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

(count up to the multiplicity      ex.  $\frac{2x}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$ )

$$\frac{2x}{x^2-6x+9} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{A(x-3) + B}{(x-3)^2}$$

$$\therefore 2x = A(x-3) + B \quad \forall x.$$

method ②  
 $x=3: 2(3) = A(\cancel{0}) + \underline{B} \quad \boxed{B=6}$

$$2x = A(x-3) + \underline{6} \quad \forall x, \text{ choose any } x\text{-value to continue}$$

$$x=1: 2(1) = A(1-3) + 6$$

$$2 = A(-2) + 6$$

$$-4 = -2A \quad \boxed{A=2}$$

$$\therefore \frac{2x}{x^2-6x+9} = \frac{2}{x-3} + \frac{6}{(x-3)^2}$$

$$\int \frac{2x}{x^2 - 6x + 9} dx = \int \frac{2}{x-3} dx + \int \frac{6}{(x-3)^2} dx$$

$u = x - 3$   
 $du = dx$

$$\int \frac{6}{u^2} du$$

$$\int 6u^{-2} du$$

$$\frac{6u^{-1}}{-1} = -\frac{6}{u}$$

$$= 2 \ln|x-3| - \frac{6}{x-3} + C$$

Notice: we could have also solved this problem using u-substitution, once we factored the

denominator:  $\int \frac{2x}{x^2 - 6x + 9} dx = \int \frac{2x}{(x-3)^2} dx$

let  $u = x - 3$   
 $du = dx$

$x = u + 3$

$$= \int \frac{2(u+3)}{u^2} du = \int \left( \frac{2u}{u^2} + \frac{6}{u^2} \right) du$$

$$= \int \left( \frac{2}{u} + 6u^{-2} \right) du = 2 \ln |u| + \frac{6u^{-1}}{-1} + C$$

$$= 2 \ln |x-3| - \frac{6}{x-3} + C. \quad \text{same answer.}$$

Ex.  $\int \frac{3x^2 + 10x + 3}{x^3 + 2x^2 + x}$

Also has a repeated linear factor.



Work on this problem  
on your own

$$x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x+1)^2$$

$$\text{So } \frac{3x^2 + 10x + 3}{x^3 + 2x^2 + x} = \frac{3x^2 + 10x + 3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}$$



$$\therefore 3x^2 + 10x + 3 = A(x+1)^2 + Bx(x+1) + Cx \quad \forall x$$

Choose convenient values for  $x$ :

$$x = -1 \quad 3(-1)^2 + 10(-1) + 3 = A(-1+1)^2 + B(-1)(-1+1) + C(-1)$$

$$3 - 10 + 3 = -C$$

$$-4 = -C \Rightarrow \boxed{C = 4}$$

$$\therefore 3x^2 + 10x + 3 = A(x+1)^2 + Bx(x+1) + 4x \quad \forall x$$

$$x = 0 \quad 3(0)^2 + 10(0) + 3 = A(0+1)^2 + B(0)(0+1) + 4(0)$$

$$\boxed{3 = A}$$

$$\therefore 3x^2 + 10x + 3 = 3(x+1)^2 + Bx(x+1) + 4x \quad \forall x$$

$$x = 1 \quad 3(1)^2 + 10(1) + 3 = 3(1+1)^2 + B(1)(1+1) + 4(1)$$

$$16 = 3(4) + B(2) + 4$$

$$0 = 2B \Rightarrow \boxed{B = 0}$$

$$\therefore \frac{3x^2 + 10x + 3}{x^3 + 2x^2 + x} = \frac{3}{x} + \frac{0}{x+1} + \frac{4}{(x+1)^2}$$

$$\text{So } \int \frac{3x^2 + 10x + 3}{x^3 + 2x^2 + x} dx = \int \frac{3}{x} dx + \int \frac{4}{(x+1)^2} dx$$

$$u = x+1 \\ du = dx$$

$$\int 4u^{-2} du = \frac{4u^{-1}}{-1} + C \\ = -\frac{4}{u} + C$$

$$= 3 \ln|x| - \frac{4}{x+1} + C.$$

$$\text{Ex. } \int \frac{2x^2 - 5x + 2}{x^3 + x} dx \quad \text{Quadratic Factor}$$

$$x^3 + x = x(x^2 + 1) \\ \leftarrow \text{can not factor further}$$

$$\text{So } \frac{2x^2 - 5x + 2}{x^3 + x} = \frac{2x^2 - 5x + 2}{x(x^2 + 1)}$$

$$= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \leftarrow \begin{array}{l} \text{linear numerator} \\ \text{for} \\ \text{quadratic factor} \end{array}$$

$$\int_0 \frac{2x^2 - 5x + 2}{x(x^2 + 1)} = \frac{A(x^2 + 1) + (Bx + C)x}{x(x^2 + 1)}$$

$$\therefore 2x^2 - 5x + 2 = A(x^2 + 1) + (Bx + C)x \quad \forall x$$

$$x=0 \quad 2(0)^2 - 5(0) + 2 = A(0^2 + 1) + (B(0) + C)0$$

$$\boxed{2 = A}$$

$$\therefore 2x^2 - 5x + 2 = 2(x^2 + 1) + (Bx + C)x$$

$$x=1 \quad 2(1)^2 - 5(1) + 2 = 2(1^2 + 1) + (B(1) + C)(1)$$

$$-1 = 4 + B + C$$

$$\underline{-5 = B + C} \Rightarrow B = -C - 5$$

$$x=2 \quad 2(2)^2 - 5(2) + 2 = 2(2^2 + 1) + (B(2) + C)2$$

$$0 = 10 + (2B + C)2$$

$$0 = 10 + 4B + 2C$$

$$\underline{-10 = 4B + 2C}$$

$$-10 = 4(-C - 5) + 2C$$

$$= -4C - 20 + 2C$$

$$10 = -2C \quad \boxed{C = -5} \Rightarrow \boxed{B = 0}$$



$$\frac{x^3 + x}{x^2 - 1} = x + \frac{2x}{x^2 - 1}$$

$$\frac{1702}{5} = 340\frac{2}{5}$$

$$\int \frac{x^3 + x}{x^2 - 1} dx = \int x dx + \underbrace{\int \frac{2x}{x^2 - 1} dx}_{u\text{-sub.}}$$

$$= \frac{x^2}{2} + \ln|x^2 - 1| + C.$$

\* after the long division, you still may need partial fractions, depends on the problem.

Ex. if degree of numerator = degree of denominator  
"the plus and minus trick"

$$\int \frac{x^2 - 2}{x^2 - 4} dx = \int \frac{x^2 - 4 + 4 - 2}{x^2 - 4} dx$$

$$= \int \frac{x^2-4}{x^2-4} dx + \int \frac{2}{x^2-4} dx$$

$$= \int 1 dx + \underbrace{\int \frac{2}{x^2-4} dx}_{\text{partial fractions...}}$$

$$x - \frac{1}{2} \ln|x+2| + \frac{1}{2} \ln|x-2| + C.$$

$$\frac{2}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2) + B(x+2)}{x^2-4}$$

$$\therefore 2 = A(x-2) + B(x+2) \quad \forall x$$

$$x=2: \quad 2 = B(4) \Rightarrow B = \frac{1}{2}$$

$$x=-2: \quad 2 = A(-4) \Rightarrow A = -\frac{1}{2}$$

$$\therefore \int \frac{2}{x^2-4} dx = -\frac{1}{2} \int \frac{1}{x+2} dx + \frac{1}{2} \int \frac{1}{x-2} dx$$

$$= -\frac{1}{2} \ln|x+2| + \frac{1}{2} \ln|x-2| + C.$$

## Integration tools:

1) Basic integration

$$\int (e^x + \sin x - \frac{3}{x^2}) dx$$

2) u-substitution for a composition of functions  
or for  $\frac{f'(x)}{g(x)}$ . (this includes trig products, lesson 11)

3) trig substitution

Always try substitutions BEFORE by parts!!

4) integration by parts for a product of functions

5) partial fractions for proper rational functions