Math 20200
Calculus II

Lesson 12
Integrals with Trigonometric Substitution

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Trig Substitution

Why we need another substitution method:

Ex. $\displaystyle\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$ if we try u-sub with

$$u = 16-x^2$$

then $du = -2x\,dx$

$\displaystyle\int \frac{-2x}{-2x\cdot \underbrace{x^2}_{x^3}\sqrt{16-x^2}}\,dx$ and $u = 16-x^2$

$$x^2 = 16-u$$

$$x^3 = \left(x^2\right)^{3/2} = \left(16-u\right)^{3/2}$$

$$= -\frac{1}{2}\int (16-u)^{-3/2}\,u^{-1/2}\,du \quad \ldots \quad \text{stuck} \ldots$$

Because of the trig identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$

we can make the following trig substitutions:

for The expression: make The substitution:

$$\sqrt{a^2-x^2}$$ $x = a\sin\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\sqrt{a^2+x^2}$$ $x = a\tan\theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\sqrt{x^2-a^2}$$ $x = a\sec\theta$ $0 \le \theta < \frac{\pi}{2}$
                                              or
                                           $\pi \le \theta < \frac{3\pi}{2}$

same as The restricted domains for finding inverses

when we have $\sqrt{a^2-x^2}$ and $x = a\sin\theta$

$$= \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$$

$$= \sqrt{a^2\cos^2\theta} = a\cos\theta \qquad \text{since } \cos\theta \ge 0 \text{ on } [-\frac{\pi}{2}, \frac{\pi}{2}]$$

gets rid of radical, easier to integrate.

(Similar simplifications with the other two trig substitutions)

Ex. $\displaystyle\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$ let $x = 4\sin\theta$
                                              then $dx = 4\cos\theta\,d\theta$

$\uparrow$
$a^2 = 16 \Rightarrow a = 4$ (always take $a > 0$)

$$= \int \frac{1}{(4\sin\theta)^2 \sqrt{16 - (4\sin\theta)^2}} \, 4\cos\theta \, d\theta$$

$$= \int \frac{1}{16\sin^2\theta \sqrt{16 - 16\sin^2\theta}} \, 4\cos\theta \, d\theta$$

$$= \int \frac{1}{16\sin^2\theta \sqrt{16(1 - \sin^2\theta)}} \, 4\cos\theta \, d\theta$$

$$= \int \frac{1}{16\sin^2\theta \sqrt{16\cos^2\theta}} \, 4\cos\theta \, d\theta$$

$$= \int \frac{1}{16\sin^2\theta \, 4\cos\theta} \, 4\cos\theta \, d\theta \qquad \text{since } \cos\theta \geq 0$$
$$\text{on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$= \int \frac{1}{16\sin^2\theta} \, d\theta = \frac{1}{16} \int \csc^2\theta \, d\theta$$

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$$\frac{d}{d\theta}(\cot\theta) = \frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{\sin\theta(-\sin\theta) - \cos\theta\cos\theta}{\sin^2\theta}$$

$$= \frac{1}{16}(-\cot\theta) + C$$

$$= \frac{-1}{\sin^2\theta} = -\csc^2\theta.$$

$x = 4\sin\theta$

$\frac{x}{4} = \sin\theta$

$\theta = \arcsin\left(\frac{x}{4}\right)$

$$= -\frac{1}{16}\underbrace{\cot\left(\arcsin\left(\frac{x}{4}\right)\right)}_{} + C$$

Simplify by using a triangle
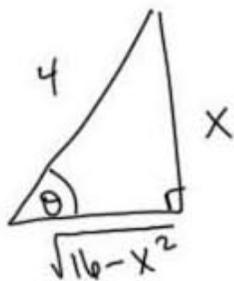
$\underset{\text{hyp}}{\overset{\text{opp}}{}} \frac{x}{4} = \sin\theta$

$$= \boxed{-\frac{1}{16} \cdot \frac{\sqrt{16-x^2}}{x} + C}$$

$$\cot\theta = \frac{adj}{opp} = \frac{\sqrt{16-x^2}}{x}$$

Ex. $\displaystyle\int x^3 \sqrt{x^2-1}\, dx$

composition try u-sub

$u = x^2 - 1 \quad\Rightarrow x^2 = u+1$

$du = 2x\, dx$

$$= \frac{1}{2}\int x^2 \underbrace{\sqrt{x^2-1}}_{\sqrt{u}}\underbrace{2x\, dx}_{du} = \frac{1}{2}\int (u+1)\sqrt{u}\, du$$

$$= \frac{1}{2}\int \left(u^{3/2} + u^{1/2}\right) du$$

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$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C = \frac{1}{5}(x^2-1)^{5/2} + \frac{1}{3}(x^2-1)^{3/2} + C.$$

OR. $\displaystyle\int x^3 \sqrt{x^2-1} \ dx$

recognize $\dfrac{}{\sqrt{x^2-a^2}}$ $a=1$

$x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$$= \int \sec^3\theta \ \sqrt{\sec^2\theta - 1} \ \sec\theta \tan\theta \, d\theta$$

$$= \int \sec^3\theta \ \sqrt{\tan^2\theta} \ \sec\theta \tan\theta \, d\theta$$

$\tan\theta \geq 0$

on $0 \leq \theta < \pi/2$

$$= \int \sec^3\theta \ \tan\theta \ \sec\theta \tan\theta \, d\theta$$

$$= \int \sec^4\theta \ \tan^2\theta \, d\theta$$

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$u = \tan\theta$
$du = \sec^2\theta \, d\theta$

$$= \int \underline{\sec^2\theta} \ \tan^2\theta \ \sec^2\theta \, d\theta$$

$$= \int (\tan^2\theta + 1) \tan^2\theta \underbrace{\sec^2\theta \, d\theta}_{du}$$

$$= \int (u^2+1) u^2 \, du$$

$$= \int (u^4 + u^2) \, du = \frac{u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{(\tan\theta)^5}{5} + \frac{(\tan\theta)^3}{3} + C$$

hyp $\frac{X}{1} = \sec\theta$
adj
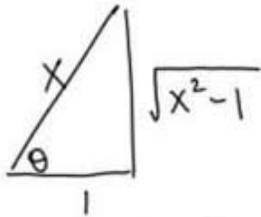
$$= \frac{\left(\sqrt{x^2-1}\right)^5}{5} + \frac{\left(\sqrt{x^2-1}\right)^3}{3} + C.$$

Same answer,

$$\tan\theta = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$

Ex. $\int \frac{1}{\sqrt{9+x^2}} \, dx$

STOP   Work on this problem
        on your own

$$\int \frac{1}{\sqrt{9+x^2}} \, dx$$

$x = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

$$= \int \frac{1}{\sqrt{9+9\tan^2\theta}} \cdot 3\sec^2\theta \, d\theta = \int \frac{3\sec^2\theta \, d\theta}{\underbrace{\sqrt{9(1+\tan^2\theta)}}_{\sec^2\theta}}$$

$$= \int \frac{\cancel{3}\sec^{\cancel{2}}\theta \, d\theta}{\cancel{3} \, \cancel{\sec\theta}} = \int \sec\theta \, d\theta =$$

$$= \ln|\underbrace{\sec\theta}_{\frac{\sqrt{9+x^2}}{3}} + \underbrace{\tan\theta}_{\frac{x}{3}}| + C$$

$x = 3\tan\theta$

$\frac{x}{3} = \tan\theta$

$\theta = \arctan(\frac{x}{3})$
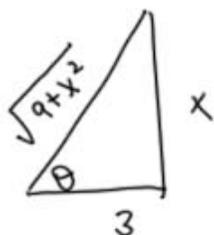
$$\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{9+x^2}}{3}$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

$$= \ln\left|\frac{\sqrt{9+x^2}+x}{3}\right| + C$$

Note: if we had $\displaystyle\int_0^4 \frac{1}{\sqrt{9+x^2}} \, dx$ definite integral,

$$= \dots \int_{0}^{\arctan 4/3} \sec\theta \, d\theta \qquad \begin{array}{l} x = 3\tan\theta \\ x = 0, \quad \theta = 0 \\ x = 4 \quad \theta = \arctan\left(\frac{4}{3}\right) \end{array}$$

$$= \ln\left|\sec\theta + \tan\theta\right|\Big]_{0}^{\arctan\frac{4}{3}}$$

$$\sec\left(\underbrace{\arctan\frac{4}{3}}_{\theta}\right) = \frac{5}{3}$$

$$= \left(\ln\left|\frac{5}{3} + \frac{4}{3}\right|\right) - \left(\ln\left|1 + 0\right|\right)$$

$$= \ln\left(\frac{9}{3}\right) = \boxed{\ln 3}$$
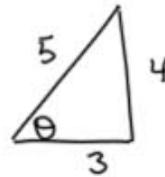
no need to go back to x.

$$\left(\text{same answer, though, } \ln\left|\frac{\sqrt{9+x^2} + x}{3}\right|\right]_{0}^{4}\right.$$

$$\left. = \ln\left|\frac{5+4}{3}\right| - \ln\left|\frac{3}{3}\right| = \ln\left(\frac{9}{3}\right) = \ln 3. \right).$$

Ex. $\displaystyle\int \frac{1}{x^2 + 4x + 7} \, dx$ doesn't look like any of the forms we can handle.

but, we can complete the square:

$$x^2 + 4x + 7 = \underbrace{x^2 + 4x + 4}_{(x+2)^2} \underbrace{-4 + 7}_{+3}$$

$$\div 2$$
$$= +2$$

So $\displaystyle\int \frac{1}{x^2 + 4x + 7}\, dx = \int \frac{1}{(x+2)^2 + 3}\, dx$

$u = x + 2$
$du = dx$

$$= \int \frac{1}{u^2 + 3}\, du = \frac{1}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C \ldots$$

one solution: $\displaystyle\frac{1}{3} \int \frac{1}{\frac{u^2}{3} + 1}\, du = \frac{1}{3} \int \frac{1}{\left(\frac{u}{\sqrt{3}}\right)^2 + 1}\, du$

then $w = \dfrac{u}{\sqrt{3}}$

$dw = \dfrac{1}{\sqrt{3}}\, du$

$$= \frac{1}{3} \cdot \frac{\sqrt{3}}{1} \int \frac{\frac{1}{\sqrt{3}}\, du}{\left(\frac{u}{\sqrt{3}}\right)^2 + 1}$$

$$= \frac{\sqrt{3}}{3} \int \frac{1}{w^2 + 1}\, dw$$

$$= \frac{\sqrt{3}}{3} \arctan w + C = \frac{\sqrt{3}}{3} \arctan\left(\frac{x+2}{\sqrt{3}}\right) + C$$

another solution method, trig sub :

$$\int \frac{1}{u^2+3}\, du$$

$u = \sqrt{3}\tan\theta$

$du = \sqrt{3}\sec^2\theta\, d\theta$

$a^2 = 3, \ a = \sqrt{3}$

$$= \int \frac{1}{3\tan^2\theta+3}\, \sqrt{3}\sec^2\theta\, d\theta \ = \ \frac{\sqrt{3}}{3}\int \frac{1}{\tan^2\theta+1}\sec^2\theta\, d\theta$$

$$= \ \frac{\sqrt{3}}{3}\int \frac{1}{\sec^2\theta}\sec^2\theta\, d\theta \ = \ \frac{\sqrt{3}}{3}\int d\theta \ = \ \frac{\sqrt{3}}{3}\theta + c$$

$$= \ \frac{\sqrt{3}}{3}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + C.$$

since $u = \sqrt{3}\tan\theta$

$$\frac{u}{\sqrt{3}} = \tan\theta$$

$$\frac{x+2}{\sqrt{3}} = \tan\theta \ \Rightarrow \ \theta = \arctan\left(\frac{x+2}{\sqrt{3}}\right)$$

Ex. $\displaystyle\int \frac{x}{\sqrt{6x-x^2-5}}\, dx$

if we try regular u-sub,

$u = 6x - x^2 - 5$

$du = (6-2x)\, dx$

↖ we don't have.

to use trig sub, we need to complete the square.

$$6x - x^2 - 5 = -x^2 + 6x - 5 = -(x^2 - 6x + 5)$$

↑ factor out the negative and complete the square inside the parentheses

$$-(x^2 - 6x + 5) = -(x^2 - 6x + 9 - 9 + 5)$$

÷2

$$= -((x-3)^2 - 4) = -(x-3)^2 + 4$$

$$= 4 - (x-3)^2$$

So $\displaystyle\int \frac{x}{\sqrt{6x - x^2 - 5}}\, dx = \int \frac{\overset{u+3}{x}}{\sqrt{4 - \underset{u}{(x-3)^2}}}\, \underset{du}{dx}$ like $\dfrac{1}{\sqrt{a^2 - x^2}}$

So $\overset{u}{x - 3} = 2\sin\theta$ + $x = 2\sin\theta + 3$ then $du = dx$

$$\underset{du}{dx} = 2\cos\theta\, d\theta$$

$$= \int \frac{(2\sin\theta + 3)(2\cos\theta\, d\theta)}{\sqrt{4 - 4\sin^2\theta}} = \int \frac{(2\sin\theta + 3)2\cos\theta\, d\theta}{\sqrt{4(1 - \sin^2\theta)}}$$

$$= \int \frac{(2\sin\theta + 3)(2\cos\theta\, d\theta)}{2\sqrt{\cos^2\theta}} = \int (2\sin\theta + 3)\, d\theta$$

$$= -2\cos\theta + 3\theta + C$$

$$= -\sqrt{4 - (x-3)^2} + 3\arcsin\left(\frac{x-3}{2}\right) + C.$$

or
$$= -\sqrt{6x - x^2 - 5} + 3\arcsin\left(\frac{x-3}{2}\right) + C.$$

$$x - 3 = 2\sin\theta$$

$$\frac{x-3}{2} = \sin\theta$$

$$\theta = \arcsin\left(\frac{x-3}{2}\right)$$
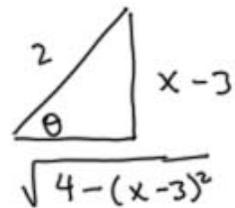
$$\cos\theta = \frac{\sqrt{4 - (x-3)^2}}{2}$$

* Notice that with each trig substitution above, a triangle was made and the substituted expression was one side of the triangle. This can help you remember the substitutions:

Ex. $\int \dfrac{1}{x^2\sqrt{16 - x^2}}\, dx$ $\sqrt{16 - x^2}$ must be one leg of a right triangle

with hypotenuse 4
and other leg x
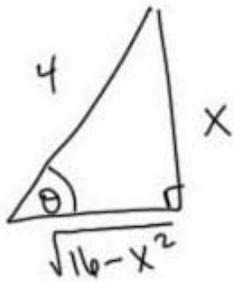
gives the substitution above,

$$\sin\theta = \frac{x}{4} \implies x = 4\sin\theta$$

could also have
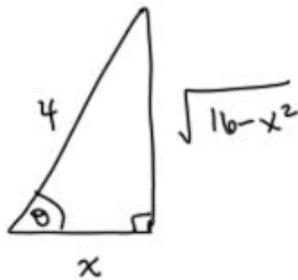
then $\cos\theta = \frac{x}{4}$

$$\implies x = 4\cos\theta$$

also a valid trig sub.

Ex. $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\, dx$

$\sqrt{9+x^2}$ would have to be the hypotenuse with legs 3 and x

$\implies \tan\theta = \dfrac{x}{3} \implies x = 3\tan\theta$