Math 20200
Calculus II

Lesson 11
Trigonometric Integrals

Dr. A. Marchese, The City College of New York

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* Key: use trig identities to transform these integrals into basic integrals or easy u-subs.

Identities:

$\sin^2 x + \cos^2 x = 1$ $\tan^2 x + 1 = \sec^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin 2x = 2 \sin x \cos x$

Also need:

$\int \sec x \, dx = \ln |\sec x + \tan x| + C.$

notice $\frac{d}{dx}\left(\ln|\sec x + \tan x| + C \right) =$

$$\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x\,(\cancel{\tan x + \sec x})}{(\cancel{\sec x + \tan x})}.$$

And:

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

$n \geq 3$

the secant reduction formula.

can find this by parts:

$u = \sec^{n-2}x \quad du = (n-2)\sec^{n-3}x\,\sec x\tan x\,dx$

$dv = \sec^2 x\,dx \qquad = (n-2)\sec^{n-2}x\tan x\,dx$

$v = \tan x$

$$\underbrace{\int \sec^n x\,dx}_{I} = \sec^{n-2}x\tan x - \int (n-2)\sec^{n-2}x\,\underbrace{\tan^2 x}_{(\sec^2 x - 1)}\,dx$$

$$= \sec^{n-2}x\tan x - \underbrace{\int (n-2)\sec^n x\,dx}_{(n-2)\,I} + \int (n-2)\sec^{n-2}x\,dx$$

$$I + (n-2)I = \sec^{n-2}x \tan x + \int (n-2) \sec^{n-2} x \, dx$$

$$(n-1)I = \sec^{n-2}x \tan x + (n-2)\int \sec^{n-2} x \, dx$$

$$I = \frac{1}{n-1} \sec^{n-2}x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx .$$

And recall:

$$\int \tan x \, dx = -\ln|\cos x| + C$$
$$= \ln|\sec x| + C$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{u} \, du =$$

$u = \cos x$

$du = -\sin x \, dx$

$$-\ln|u| + C =$$

$$= -\ln|\cos x| + C$$

$$= \ln|(\cos x)^{-1}| + C$$ ← exponent, not inverse

$$= \ln|\sec x| + C .$$

We've highlighted all the tools we need to solve trig integrals.

Ex. $\int \sin^6 x \cos^3 x \, dx$

we see compositions

$(\sin x)^6 \quad (\cos x)^3$

if we let $u = \sin x$,
then $du = \cos x \, dx$

and we have an extra $\cos^2 x$

so we write $\cos^2 x = 1 - \sin^2 x$

$$\int \sin^6 x \cos^3 x \, dx = \int \underbrace{\sin^6 x}_{u^6} \underset{\substack{\uparrow \\ \text{trig} \\ \text{ident}}}{\cos^2 x} \underbrace{\cos x \, dx}_{du}$$

$$= \int \underbrace{\sin^6 x}_{u^6} \underbrace{(1 - \sin^2 x)}_{(1-u^2)} \underbrace{\cos x \, dx}_{du}$$

$$= \int u^6 (1 - u^2) \, du = \int (u^6 - u^8) \, du$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + C = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C.$$

If we had let $u = \cos x$, $du = -\sin x \, dx$

and we'd have extra $\sin^5 x$ ← since The power is odd, we can't replace with cosines.

But consider $\int \sin^5 x \cos^3 x \, dx$

can let $u = \sin x$ as above,

and get $\int u^5(1-u^2)\,du = \int (u^5 - u^7)\,du$

$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$$

OR

let $u = \cos x \quad du = -\sin x \, dx$

then $\int \sin^5 x \cos^3 x \, dx = -\int \sin^4 x \cos^3 x (-\sin x)\,dx$

$$= -\int (\sin^2 x)^2 \cos^3 x \, (-\sin x)\,dx$$

$$= -\int (1-\cos^2 x)^2 \cos^3 x \, (-\sin x)\,dx$$

$$= -\int (1-u^2)^2 u^3 \, du =$$

$$= -\int (1-2u^2+u^4)u^3\,du$$

$$= -\int \left(u^3 - 2u^5 + u^7\right) du = -\left(\frac{u^4}{4} - \frac{2u^6}{6} + \frac{u^8}{8}\right) + K$$

$$= -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{3} - \frac{\cos^8 x}{8} + K$$

(can see by graphing $y = -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{3} - \frac{\cos^8 x}{8}$ and $y = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8}$ differ by a constant).

Ex. $\int \frac{\sin^5 2x}{\sqrt{\cos 2x}}\,dx$ notice $2x$ in $\underline{\underline{\text{both}}}$ so ok.

$= \int (\sin 2x)^5 (\cos 2x)^{-1/2}\,dx$ must let $u = \cos 2x$

$du = -2\sin 2x\,dx$

$$= -\frac{1}{2}\int (\sin 2x)^4 (\cos 2x)^{-1/2}(-2)\sin 2x\,dx$$

$$= -\frac{1}{2}\int (\sin^2 2x)^2 (\cos 2x)^{-1/2}(-2)\sin 2x\,dx$$

$$= -\frac{1}{2}\int (1-\cos^2 2x)^2 (\cos 2x)^{-1/2}(-2)\sin 2x\,dx$$

$$= -\frac{1}{2}\int (1-u^2)^2 u^{-1/2}\,du$$

$$= -\frac{1}{2}\int (1-2u^2+u^4)u^{-1/2}\,du$$

$$= -\frac{1}{2}\int (u^{-1/2}-2u^{3/2}+u^{7/2})\,du$$

$$= -\frac{1}{2}\left(\frac{u^{1/2}}{\frac{1}{2}} - \frac{2u^{5/2}}{5/2} + \frac{u^{9/2}}{\frac{9}{2}}\right) + C$$

$$= -\frac{1}{2}\cdot\frac{2}{1}u^{1/2} + \frac{1}{2}\cdot\frac{2}{1}\cdot\frac{2}{5}u^{5/2} - \frac{1}{2}\cdot\frac{2}{9}u^{9/2} + C.$$

$$= -(\cos x)^{1/2} + \frac{2}{5}(\cos x)^{5/2} - \frac{1}{9}(\cos x)^{9/2} + C.$$

Ex. $\int \sin^2 x\,dx$ we see $(\sin x)^2$

if we let $u = \sin x$

$du = \cos x\,dx$

$\uparrow$

we don't have

Keep in mind $$\underset{\substack{\uparrow \\ \text{can't integrate} \\ \text{on its own}}}{\sin^2 x} = \underset{\substack{\uparrow \\ \underline{\underline{\text{can}}} \text{ integrate on} \\ \text{its own}}}{\frac{1}{2}(1-\cos 2x)}$$

so $$\int \sin^2 x \, dx = \int \frac{1}{2}(1-\cos 2x)\, dx =$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

Ex. $\displaystyle\int_0^{\pi/2} \sin^4 x \, dx$ we have $(\sin x)^4$

again, if we let $u = \sin x$

we don't have $du = \cos x \, dx$

$$\int_0^{\pi/2} \sin^4 x \, dx = \int_0^{\pi/2} (\sin^2 x)^2 \, dx = \int_0^{\pi/2} \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$$

$$= \int_0^{\pi/2} \frac{1}{4}\left(1 - 2\cos 2x + \overbrace{\cos^2 \underline{\underline{2x}}}^{\text{still a problem}}\right) dx$$

use $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$

$$= \int_0^{\pi/2} \frac{1}{4}\left(1 - 2\cos 2x + \underbrace{\frac{1}{2}(1+\cos 4x)}_{\frac{1}{2}+\frac{1}{2}\cos 4x}\right) dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x - 2\,\frac{1}{2}\sin(2x) + \frac{1}{2}\cdot\frac{1}{4}\sin(4x)\right]_0^{\pi/2}$$

$$= \left[\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right]_0^{\pi/2}$$

$$= \left(\frac{3}{8}\cdot\frac{\pi}{2} - 0 + 0\right) - \left(0 - 0 + 0\right)$$

$$= \boxed{\frac{3\pi}{16}}$$

Ex. $\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$ if we let $u = 1+\cos 4x$

$du = -4\sin 4x\, dx$

$\uparrow$

we don't have

but we have $1+\cos 4x = 2\cos^2 2x$

comes from $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$$

$$2\cos^2 2x = 1+\cos 4x$$

So $\displaystyle\int_0^{\pi/4}\sqrt{1+\cos 4x}\,dx = \int_0^{\pi/4}\sqrt{2\cos^2 2x}\,dx$

$\displaystyle= \int_0^{\pi/4}\sqrt{2}\,\sqrt{(\cos 2x)^2}\,dx$ since $\cos 2x \geq 0$ on $\Big[\underset{\substack{\uparrow\\ \cos(0)}}{0}, \underset{\substack{\uparrow\\ \cos(\frac{2\pi}{4})=\cos(\frac{\pi}{2})}}{\frac{\pi}{4}}\Big]$,

$\displaystyle= \int_0^{\pi/4}\sqrt{2}\cos 2x\,dx$

$\displaystyle= \sqrt{2}\,\frac{1}{2}\sin 2x\Big|_0^{\pi/4}$

$$= \frac{\sqrt{2}}{2}\left[\sin\left(\frac{2\pi}{4}\right)-\sin(2\cdot 0)\right] = \frac{\sqrt{2}}{2}\left[\sin\left(\frac{\pi}{2}\right)-\sin 0\right]$$

$$= \frac{\sqrt{2}}{2}(1-0) = \boxed{\frac{\sqrt{2}}{2}}$$

Ex. $\int \tan^3 x \sec x \, dx$

we see $(\tan x)^3$

if $u = \tan x$,

$du = \sec^2 x \, dx$

↑ we dont have This

so we try $u = \sec x$ then $du = \sec x \tan x \, dx$

$$\int \tan^3 x \sec x \, dx = \int \tan^2 x \underbrace{\sec x \tan x \, dx}_{du}$$

$$= \int \underbrace{(\sec^2 x - 1)}_{(u^2-1)} \underbrace{\sec x \tan x \, dx}_{du}$$

$$= \int (u^2 - 1) \, du = \frac{u^3}{3} - u + C$$

$$= \frac{\sec^3 x}{3} - \sec x + C.$$

Ex. $\int \tan^2 x \sec^4 x \, dx$

we see $(\tan x)^2$

$(\sec x)^4$

if we let $u = \tan x$

$du = \sec^2 x \, dx$

and the extra $\sec^2 x = \tan^2 x + 1$

~~if we let $u = \sec x$~~

~~$du = \sec x \tan x \, dx$~~

~~then we have an~~

~~extra $\tan x$ ...~~

~~won't work.~~

$$\int \tan^2 x \sec^4 x \, dx = \int \underbrace{\tan^2 x}_{u^2} \underset{\substack{\uparrow \\ \text{trig identity}}}{\sec^2 x} \underbrace{\sec^2 x \, dx}_{du}$$

$$= \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, dx$$

$$= \int u^2 (u^2 + 1) \, du = \int (u^4 + u^2) \, du$$

$$= \frac{u^5}{5} + \frac{u^3}{3} + C = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C.$$

Ex. $\int \tan^2 x \sec^3 x \, dx$ we see $(\tan x)^2$, $(\sec x)^3$

if we let $u = \tan x$

$du = \sec^2 x \, dx$

but then we have an

extra $\sec x$.

won't work.

if we let $u = \sec x$

$du = \sec x \tan x \, dx$

then we have an

extra $\tan x$.

won't work.

here we'll make use of The secant reduction formula by getting This all in terms of sec x.

$$\int \tan^2 x \sec^3 x \, dx$$

$$= \int (\sec^2 x - 1)\sec^3 x \, dx = \int (\sec^5 x - \sec^3 x)\, dx$$

$$= \underbrace{\int \sec^5 x \, dx}_{\text{by secant reduction formula}} - \int \sec^3 x \, dx$$

$$= \frac{1}{4} \sec^3 x \tan x + \underbrace{\frac{3}{4} \int \sec^3 x \, dx - \int \sec^3 x \, dx}_{}$$

$$= \frac{1}{4} \sec^3 x \tan x - \frac{1}{4} \underbrace{\int \sec^3 x \, dx}_{}$$

$$= \frac{1}{4} \sec^3 x \tan x - \frac{1}{4} \overbrace{\left[\frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx \right]}^{\text{secant reduction formula}}$$

$$= \frac{1}{4} \sec^3 x \tan x - \frac{1}{8} \sec x \tan x - \frac{1}{8} \int \sec x \, dx$$

$$= \frac{1}{4} \sec^3 x \tan x - \frac{1}{8} \sec x \tan x - \frac{1}{8} \ln|\sec x + \tan x| + C$$