Math 20200
Calculus II

Lesson 10
Integration by Parts

Dr. A. Marchese, The City College of New York

Table of Contents:

1. Basic integrals	00:10	p.2
2. Integration by parts formula	02:29	p.3
3. LIATE for choosing u	06:07	p.5
4. Repeated by parts	11:58	p.8
5. Circular by parts	18:15	p.11

Integration By Parts

Recall, Basic Integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

When solving an integral, first look to see if it is a basic integral.
if not, look for a composition or a $\frac{g'}{g}$ situation (and use u-sub).

If it is neither basic nor u-sub, look for a product of functions and use integration by parts:

We know from the product rule,

$$
\begin{aligned}
u'v + uv' &= (uv)' \\
-u'v \quad & \quad\; -u'v
\end{aligned}
$$

$$uv' = (uv)' - u'v$$

$$\int u\underbrace{v'dx}_{dv} = \int (uv)'dx - \int \underline{u'}\, v\, \underline{dx} \qquad u'dx = du$$

$$\int u\,\underline{dv} = uv - \int v\,\underline{du}$$

Integration By Parts $\boxed{\int u\,dv = uv - \int v\,du}$

Ex. $\int \underbrace{x}_{u} \underbrace{e^x \, dx}_{dv}$

① basic integral? no.
② u-sub? composition? no
③ product? yes $x \cdot e^x$

let $u = x \rightarrow du = dx$
$dv = e^x dx \Rightarrow v = e^x$

then $\int \underbrace{x}_{u} \underbrace{e^x \, dx}_{dv} = \underbrace{x}_{u} \cdot \underbrace{e^x}_{v} - \overbrace{\int \underbrace{e^x}_{v} \underbrace{dx}_{du}}^{\text{easier to solve than original integral.}}$

$= xe^x - e^x + C.$

What if we chose $u = e^x$ & $dv = x\,dx$

$\int x \overbrace{e^x}^{u} dx = \underbrace{e^x}_{u} \cdot \underbrace{\frac{x^2}{2}}_{v} - \underbrace{\int \underbrace{\frac{x^2}{2}}_{v} \underbrace{e^x dx}_{du}}_{\text{harder to solve than original integral.}}$

$u = e^x \rightarrow du = e^x dx$
$dv = x\,dx \Rightarrow v = \frac{x^2}{2}$

wrong choice

How to choose $\underline{\underline{u}}$ for integration by parts:

L ogarithms
I nverse trig function
A lgebraic $\quad x^2,\ x^3+1$
T rig functions
E xponential function.

$$\int u\,dv = uv - \int v\,du$$

Ex. $\displaystyle\int x^2 \ln x \, dx$

$u = \ln x \;\rightarrow\; du = \frac{1}{x}\,dx$

$dv = x^2 dx \qquad v = \frac{x^3}{3}$

$$\int_1^2 x^2 \underbrace{\ln x}_{u}\, dx = \underbrace{(\ln x)}_{u}\underbrace{\frac{x^3}{3}}_{v}\Bigg|_1^2 - \underbrace{\int_1^2 \underbrace{\frac{x^3}{3}}_{v}\cdot\underbrace{\frac{1}{x}\,dx}_{du}}_{\text{SIMPLIFY FIRST}}$$

$$= \frac{x^3}{3}\ln x\Bigg|_1^2 - \frac{1}{3}\int_1^2 x^2\,dx$$

$$= \left[\frac{x^3}{3}\ln x - \frac{1}{3}\frac{x^3}{3}\right]_1^2 \cancel{+C}$$

$$= \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^2 \cancel{+C}$$

$$\left(\frac{2^3}{3}\ln 2 - \frac{2^3}{9}\right) - \left(\cancel{\frac{1^3}{3}\underbrace{\ln(1)}_{0}} - \frac{1^3}{9}\right)$$

$$\frac{8}{3}\ln 2 - \frac{8}{9} + \frac{1}{9}$$

$$\frac{8}{3}\ln 2 - \frac{7}{9}.$$

Ex. $\int \underbrace{\arcsin x}_{u}\,\underbrace{dx}_{dv}$ tricky not basic
not composition
doesn't look like a product

but treat like product

$u = \arcsin x \rightarrow du = \frac{1}{\sqrt{1-x^2}}dx$

$dv = dx \rightarrow v = x$

$$\int \arcsin x \, dx = (\arcsin x)(x) - \frac{1}{-2}\int \frac{-2x}{\sqrt{1-x^2}}\, dx$$

$w = 1-x^2$

$dw = -2x\,dx$

$$= x \arcsin x + \frac{1}{2}\int w^{-1/2}\, dw$$

$$= x \cdot \arcsin x + \frac{1}{2}\,\frac{w^{1/2}}{\frac{1}{2}} + C$$

$$= x \arcsin x + \sqrt{1-x^2} + C.$$

Repeated by parts:

Ex $\int x^2 e^x \, dx$

LIATE

$u = x^2 \Rightarrow du = 2x\,dx$

$dv = e^x dx \Rightarrow v = e^x$

$$\int x^2 e^x dx = x^2 e^x - \int e^x (2x) dx$$

$$= x^2 e^x - 2 \underbrace{\int x e^x dx}_{\text{by parts}}$$

$$= x^2 e^x - 2 \left[x e^x - e^x \right] + C$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

$u = x \quad du = dx$

$dv = e^x dx \quad v = e^x$

$$= x^2 e^x - 2x e^x + 2e^x + C.$$

OR $= e^x (x^2 - 2x + 2) + C.$

trick for repeated integration by parts when the derivative of u is eventually $= 0$.
(u algebraic with derivative eventually $= 0$)

	u and its derivatives	dv and its antiderivatives
$+$	x^2	e^x
$-$	$2x$	e^x
$+$	2	e^x
$-$	0	e^x

$$\int x^2 e^x\,dx = x^2e^x - 2xe^x + 2e^x + C.$$

Ex. $\int x^3 \cos x \, dx$

Work on this problem on your own

$$\int x^3 \cos x \, dx$$

LIATE

$\uparrow$

$u = x^3 \quad du = 3x^2 dx$

$dv = \cos x \, dx \quad v = \sin x$

$$= x^3 \sin x - \int \underbrace{3x^2 \sin x \, dx}_{\text{by parts}}$$

$\hat{u} = x^2 \quad d\hat{u} = 2x \, dx$

$d\hat{v} = \sin x \, dx \quad \hat{v} = -\cos x$

$$= x^3 \sin x - 3\left[\underline{x^2(-\cos x) + \int 2x \cos x \, dx}\right]$$

$$= x^3 \sin x + 3x^2 \cos x - 6 \int \underbrace{x \cos x \, dx}_{\text{by parts}}$$

$\tilde{u} = x \quad d\tilde{u} = dx$

$d\tilde{v} = \cos x \, dx \quad \tilde{v} = \sin x$

$$= x^3 \sin x + 3x^2 \cos x - 6\left[\underline{x \sin x - \int \sin x \, dx}\right]$$

$$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C.$$

OR by table, since $u = x^3$ and $dv = \cos x \, dx$:

	u and its derivatives	dv and its antiderivatives
$+$	x^3	$\cos x$
$-$	$3x^2$	$\sin x$
$+$	$6x$	$-\cos x$
$-$	6	$-\sin x$
$+$	0	$\cos x$

$$\therefore \int x^3 \cos x\, dx = x^3 \sin x - 3x^2(-\cos x) + 6x(-\sin x) + -6\cos x + c.$$

$$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + C.$$

Circular by Parts:

$$\int e^{2x} \sin x\, dx$$

LIATE

$u = \sin x \Rightarrow du = \cos x\, dx$

$dv = e^{2x}\, dx \quad v = \frac{1}{2}e^{2x}$

$$\int e^{2x} \sin x\, dx = \frac{1}{2}e^{2x}\sin x - \underbrace{\int \frac{1}{2} e^{2x} \cos x\, dx}_{\text{another by parts}}$$

$\hat{u} = \cos x$
$d\hat{u} = -\sin x\, dx$
$d\hat{v} = e^{2x} dx$
$\hat{v} = \frac{1}{2}e^{2x}$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left[\underbrace{\tfrac{1}{2}e^{2x}}_{\hat{v}} \underbrace{\cos x}_{\hat{u}} \underset{-}{+} \int \underbrace{\tfrac{1}{2}e^{2x}}_{\hat{v}} \underbrace{\sin x\, dx}_{d\hat{u}} \right]$$

$$\underbrace{\int e^{2x}\sin x\, dx}_{I} = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\underbrace{\int e^{2x}\sin x\, dx}_{I + c}$$

$$I = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}(I + c)$$

$$I + \frac{1}{4}I = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}C$$

$$\frac{5}{4}I = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}C$$

$$I = \frac{4}{5}\left(\frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}C\right)$$

$$= \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + K$$

$$\therefore \int e^{2x}\sin x\,dx = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + K.$$