

Indeterminate forms and L'Hospital's Rule

Recall: $\lim_{x \rightarrow 2} \frac{x^2 - 4}{2 - x}$

try plugging in
 $\frac{0}{0}$ ← an indeterminate form, means we need more info.

so we factor:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{2 - x} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{2-x} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{-(x-2)}$$

$$= \lim_{x \rightarrow 2} \frac{x+2}{-1} = \frac{2+2}{-1} = -4.$$

Also recall: $\lim_{x \rightarrow \infty} \frac{5x^2 - 4}{x - x^2}$ $\frac{\infty}{\infty}$ another indeterminate form, means we need more info

so we divide by the highest power we see in the denominator, x^2 :

$$\lim_{x \rightarrow \infty} \frac{5x^2 - 4}{x - x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{5x^2}{x^2} - \frac{4}{x^2}}{\frac{x}{x^2} - \frac{x^2}{x^2}} = \lim_{x \rightarrow \infty} \frac{5 - \frac{4}{x^2}}{\frac{1}{x} - 1}$$

we know $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ $\therefore \lim_{x \rightarrow \infty} \frac{5}{-1} = -5$.

$\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$

Recall, for rational functions:

highest powers same in numerator and denominator,

$\lim_{x \rightarrow \pm\infty} = \text{ratio of coefficients}$ $\lim_{x \rightarrow \infty} \frac{5x^2 - 4}{x^2} = -5$

highest power greater in denominator, $\lim_{x \rightarrow \infty} \frac{5x - 1}{x^3 + x^2} = 0$

$\lim_{x \rightarrow \pm\infty} = 0$

highest power greater in numerator,

$\lim_{x \rightarrow \pm\infty} = \pm\infty$ (must determine from problem)

Ex $\lim_{x \rightarrow \infty} \frac{5x^2 - 4}{1 - x} = \lim_{x \rightarrow \infty} \frac{5x^2 - 4}{1 - x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{5x - \frac{4}{x}}{\frac{1}{x} - 1} = -\infty$

But what about $\lim_{x \rightarrow 0} \frac{\sin x}{e^{4x} - 1}$ try plugging in $\frac{\sin(0)}{e^0 - 1} = \frac{0}{0}$ indeterminate form. need more info.

and $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$ indeterminate form.

L'Hospital's Rule will help determine these limits.

L'Hospital's Rule only for limits with

indeterminate forms of type $\boxed{\frac{0}{0} \text{ or } \frac{\infty}{\infty}}$:

For f & g differentiable with $g'(x) \neq 0$ near $x=c$,

If $\lim_{x \rightarrow c} f(x) = 0$ and $\lim_{x \rightarrow c} g(x) = 0$, OR

$\lim_{x \rightarrow c} f(x) = \pm\infty$ and $\lim_{x \rightarrow c} g(x) = \pm\infty$, then:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} \quad (\text{here, } c \text{ can be } \pm\infty).$$

Revisiting The examples above:

$$\frac{0}{0} \quad \lim_{x \rightarrow 2} \frac{x^2 - 4}{2 - x} \stackrel{\text{LH}}{=} \lim_{x \rightarrow 2} \frac{2x}{-1} = \frac{2(2)}{-1} = -4.$$

$$\frac{\infty}{-\infty} \quad \lim_{x \rightarrow \infty} \frac{5x^2 - 4}{x - x^2} \stackrel{\text{LH}}{=} \lim_{x \rightarrow \infty} \frac{\overset{\infty}{10x}}{\underset{-\infty}{1 - 2x}} \stackrel{\text{LH}}{=} \lim_{x \rightarrow \infty} \frac{10}{-2} = -5.$$

Now we can solve: $\lim_{x \rightarrow 0} \frac{\sin x}{e^{4x} - 1}$

$$\frac{0}{0} \quad \lim_{x \rightarrow 0} \frac{\sin x}{e^{4x} - 1} \stackrel{\text{LH}}{=} \lim_{x \rightarrow 0} \frac{\cos x}{4e^{4x}} = \frac{\cos(0)}{4e^{4(0)}} = \frac{1}{4}$$

And: $\lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{\text{LH}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0$

Ex. $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

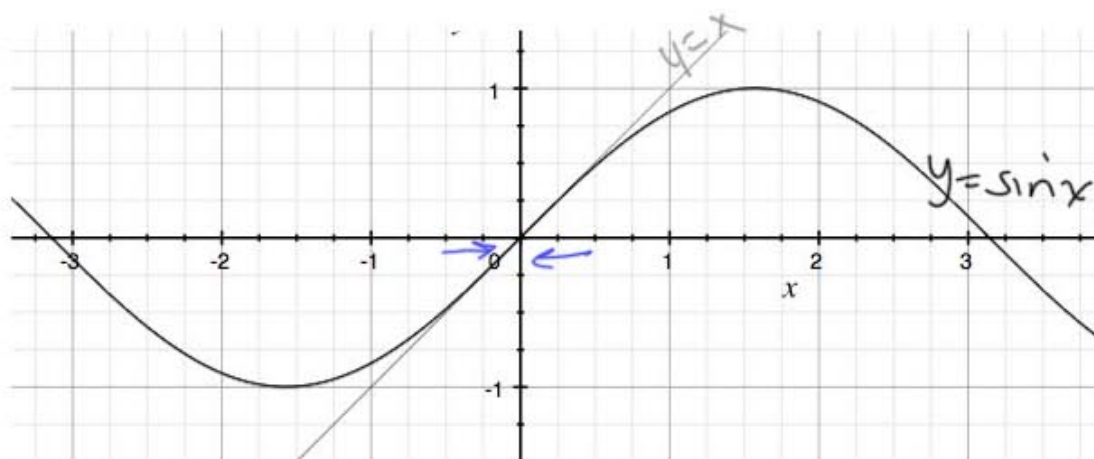


Work on this problem on your own

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad \frac{0}{0}$$

$$\stackrel{\text{LH}}{=} \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos(0) = 1$$

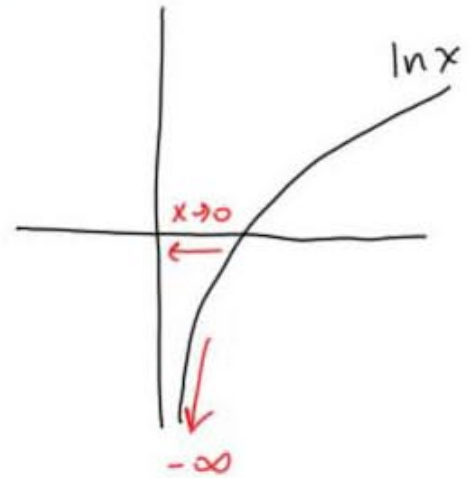
and $\sin x \rightarrow 0$
and $x \rightarrow 0$ at
the same
rate



$$\lim_{x \rightarrow \infty} \frac{1 - e^{-5x}}{2x} = \frac{1}{\infty} = 0 \quad \text{not L'Hospital's Rule.}$$

Ex. $\lim_{x \rightarrow 0^+} x^2 \ln x$ (0)(-\infty) another indeterminate form

rewrite $\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^2}} = \frac{-\infty}{+\infty}$



$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} \quad x^{-2} \quad -2x^{-3}$

$\stackrel{\text{simplify}}{=} \lim_{x \rightarrow 0^+} -\frac{1}{x} \cdot \frac{x^3}{2}$

$= \lim_{x \rightarrow 0^+} -\frac{1}{2} x^2 = 0.$

Indeterminate Forms

$$\frac{0}{0} \text{ LH directly}$$

$$\frac{\infty}{\infty} \text{ LH directly}$$

$0(\infty)$ change to $\frac{0}{0}$ or $\frac{\infty}{\infty}$
then use L'Hospital.

$$\infty - \infty$$

get it out of subtraction form.
common denominator, factoring, use exponentials

$$0^0$$

$$\infty^0$$

$$1^\infty$$

use the natural log to help find limit

$$\text{Ex. } \lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

$$\frac{1}{0^+} = +\infty$$

$$\frac{1}{0^-} = -\infty$$

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

$0^+ \quad 0^+$

$$(\infty - \infty)$$

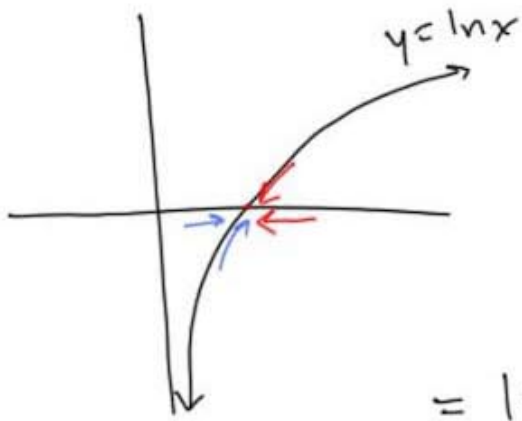
$$\lim_{x \rightarrow 1^-} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$$

$0^- \quad 0^-$

$$(-\infty - (-\infty))$$

$$(-\infty + \infty)$$

↑
indeterminate form ↑



make common denominator

$$= \lim_{x \rightarrow 1} \frac{x-1 - \ln x}{(\ln x)(x-1)}$$

plugging in
 $x=1$

$$\frac{1-1 - \ln 1}{0(0)} = \frac{0}{0}$$

$$\underline{\underline{\text{LH}}} \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + \ln x}$$

$$= \lim_{x \rightarrow 1} \frac{\frac{x-1}{x}}{\frac{x-1+x \ln x}{x}} = \lim_{x \rightarrow 1} \frac{x-1}{x-1+x \ln x} \frac{0}{0}$$

$$\underline{\underline{\text{LH}}} \lim_{x \rightarrow 1} \frac{1}{1 + \ln x + x \cdot \frac{1}{x}}$$

$$= \lim_{x \rightarrow 1} \frac{1}{2 + \ln x} = \frac{1}{2}$$

Ex. $\lim_{x \rightarrow \infty} (\ln x - x)$
 $\infty - \infty$

$$y = \lim_{x \rightarrow \infty} (\ln x - x)$$

take exponential
of both sides

$$e^y = e^{\lim_{x \rightarrow \infty} (\ln x - x)}$$

$$e^y = \lim_{x \rightarrow \infty} e^{\ln x - x}$$

by continuity of $y = e^x$

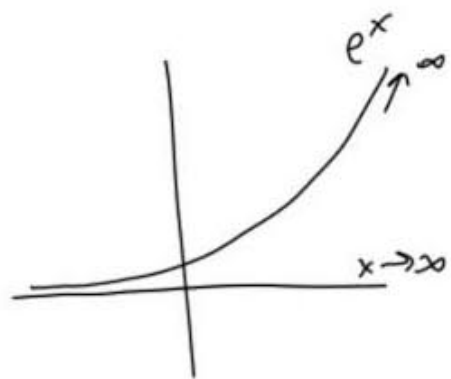
$$= \lim_{x \rightarrow \infty} e^{\ln x} e^{-x}$$

$$e^y = \lim_{x \rightarrow \infty} x e^{-x}$$

$\infty \cdot 0$

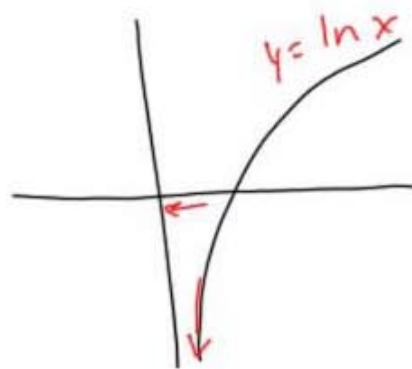
$$e^y = \lim_{x \rightarrow \infty} \frac{x}{e^x} \quad \frac{\infty}{\infty}$$

$$e^y \stackrel{\text{LH}}{=} \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0^+$$



$$e^y = 0^+$$

$$y = \ln(0^+) = -\infty$$



OR

$$\lim_{x \rightarrow \infty} (\ln x - x) = \lim_{x \rightarrow \infty} x \left(\frac{\ln x}{x} - 1 \right) = \infty(-1) = -\infty.$$

We know $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0$ (above).

Ex. $\lim_{x \rightarrow 0^+} (\cos x)^{1/x}$ *x in base* *x in exponent* *use natural log!*

$$y = \lim_{x \rightarrow 0^+} (\cos x)^{1/x}$$

1^∞ indeterminate form.

$$\ln y = \ln \left(\lim_{x \rightarrow 0^+} (\cos x)^{1/x} \right)$$

$$\ln y = \lim_{x \rightarrow 0^+} \ln (\cos x)^{1/x} \quad \text{by continuity}$$

$$\ln y = \lim_{x \rightarrow 0^+} \left(\frac{1}{x} \ln(\cos x) \right)$$

+∞ *ln(1)*
0

$$\ln y = \lim_{x \rightarrow 0^+} \frac{\ln(\cos x)}{x} \quad \frac{0}{0}$$

$$\text{LH} = \lim_{x \rightarrow 0^+} \frac{-\frac{\sin x}{\cos x}}{1} = -\tan(0) = 0.$$

$$\begin{aligned} \therefore \ln y = 0 &\Rightarrow \frac{e^{\ln y}}{y} = e^0 = 1 \\ e^0 = y &= \boxed{1}. \end{aligned}$$