Math 20200
Calculus II

Lesson 8
Hyperbolic Functions

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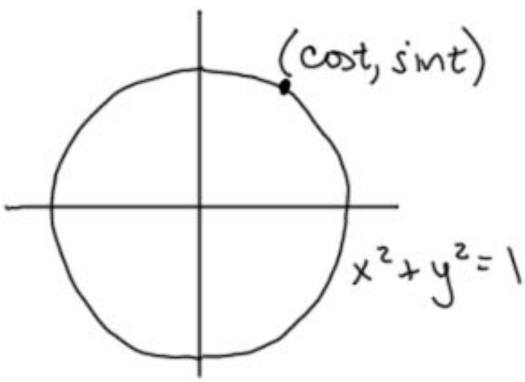
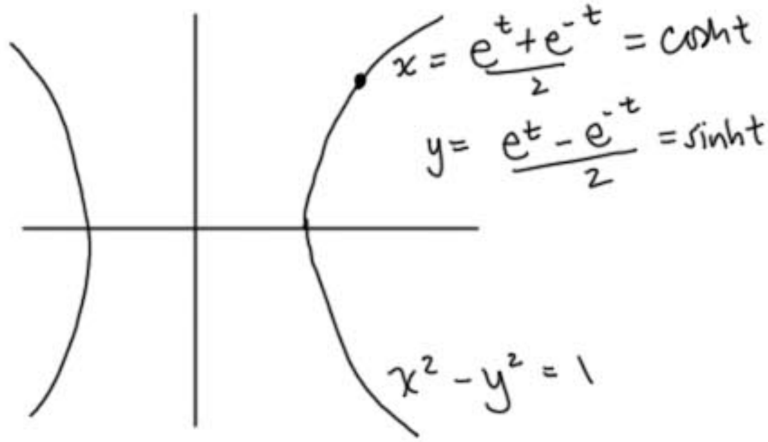
Hyperbolic Functions

Consider $\dfrac{e^x - e^{-x}}{2}$ and $\dfrac{e^x + e^{-x}}{2}$.

These functions have the same relationship to the hyperbola $x^2 - y^2 = 1$ as $y = \cos x$ + $y = \sin x$ have with the unit circle $x^2 + y^2 = 1$.





$\therefore \cos^2 t + \sin^2 t = 1$

call $\dfrac{e^t + e^{-t}}{2} = \cosh t$

$\dfrac{e^t - e^{-t}}{2} = \sinh t$

hyperbolic cosine +

hyperbolic sine

$\therefore \cosh^2 t - \sinh^2 t = 1$.

Ex. Evaluate $\sinh(3)$

$$\sinh(3) = \frac{e^{3} - e^{-3}}{2}$$

Ex. Evaluate $\cosh(\ln 3)$

$$\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2} =$$

Work on this problem
on your own

$$e^{-\ln 3} = e^{\ln(3^{-1})} = e^{\ln(\frac{1}{3})} = \frac{1}{3}$$

$$= \frac{3 + (e^{\ln 3})^{(-1)}}{2} = \frac{3 + 3^{-1}}{2} = \frac{3 + \frac{1}{3}}{2}$$

$$= \frac{\frac{10}{3}}{2} = \frac{10}{6} = \boxed{\frac{5}{3}}$$

Ex. $\sinh(0) = \dfrac{e^{0} - e^{-0}}{2} = \dfrac{1-1}{2} = 0$

$\cosh(0) = \dfrac{e^{0} + e^{-0}}{2} = \dfrac{1+1}{2} = \dfrac{2}{2} = 1.$

Ex. prove $\sinh(-x) = -\sinh(x)$.

$$\sinh(-x) = \frac{e^{(-x)} - e^{-(-x)}}{2}$$

$$= \frac{e^{-x} - e^{x}}{2} = -\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$= -\sinh x.$$

So we have: $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$, $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

and we define $\tanh x = \dfrac{\sinh x}{\cosh x}$, $\coth x = \dfrac{\cosh x}{\sinh x}$

$\operatorname{sech} x = \dfrac{1}{\cosh x}$ and $\operatorname{csch} x = \dfrac{1}{\sinh x}$

Ex. If $\sinh(x) = \frac{3}{4}$ find the values of $\cosh(x)$ and $\tanh(x)$.

to find $\cosh x$, use $\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x - \left(\frac{3}{4}\right)^2 = 1$$

$$\cosh^2 x - \frac{9}{16} = 1$$

$$\cosh^2 x = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\cosh x = \pm\sqrt{\frac{25}{16}}$$

$$= \pm \frac{5}{4}$$

cosh must be positive

$$\frac{e^x + e^{-x}}{2}$$

then $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{\frac{3}{4}}{\frac{5}{4}} = \boxed{\dfrac{3}{5}}$.

Derivatives of Hyperbolic Functions:

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right)$$

$$= \frac{1}{2}\left(e^x - e^{-x}(-1)\right) = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$$

Similarly, $\frac{d}{dx}(\cosh x) = \sinh x$ (exercise)

Ex. $f(t) = \ln(\sinh t)$

$$f'(t) = \frac{1}{\sinh t} \cdot \frac{d}{dt}(\sinh t) = \frac{\cosh t}{\sinh t}$$

$$= \coth(t).$$

Ex. $y = \sinh(\cosh x)$

$$y' = \cosh(\cosh x) \cdot \frac{d}{dx}(\cosh x)$$ chain rule

$$= \cosh(\cosh x) \cdot \sinh x$$

$$= \sinh x \cdot \cosh(\cosh x).$$

Ex. $\int \frac{\sinh x}{1 + \cosh^2 x}\, dx$

Work on this problem on your own

$$\int \frac{\sinh x}{1+\cosh^2 x}\,dx = \int \frac{\sinh x}{1+(\cosh x)^2}\,dx \qquad \begin{array}{l} u = \cosh x \\ du = \sinh x\,dx \end{array}$$

$$= \int \frac{1}{1+u^2}\,du = \arctan u + C$$

$$= \arctan(\cosh x) + C.$$

Since $\frac{d}{dx}(\sinh x) = \cosh x > 0 \ \forall x$, we know

$\sinh x$ is one to one and hence has an inverse.

Ex. Prove that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

2 ways we can do this:

1) start with $y = \sinh x$ and find the inverse function

2) let $f(x) = \sinh x$ and $g(x) = \ln\left(x + \sqrt{x^2+1}\right)$

and show $f(g(x)) = x$ and $g(f(x)) = x \ \forall x \in \mathbb{R}$

1) $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$ solve for x

$2y = e^x - e^{-x}$ mult. by e^x

$2ye^x = e^{2x} - 1$

$0 = e^{2x} - 2ye^x - 1$ treat as a quadratic

$0 = (e^x)^2 - 2ye^x - 1$ $a=1$ $b=-2y$ $c=-1$

$$e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$$

notice e^x must be positive, so we

can't have $y - \sqrt{y^2+1}$ $\left(\sqrt{y^2+1} > y\right)$

$\therefore e^x = y + \sqrt{y^2+1}$

$x = \ln\left(y + \sqrt{y^2+1}\right)$ now switch x + y

$y = \ln\left(x + \sqrt{x^2+1}\right) = \sinh^{-1} x$.

2) $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$ $g(x) = \ln\left(x + \sqrt{x^2+1}\right)$

$$f(g(x)) = \frac{e^{\ln(x+\sqrt{x^2+1})} - e^{-\ln((x+\sqrt{x^2+1}))}}{2} =$$

$$= \frac{\left(x+\sqrt{x^2+1}\right) - e^{\ln\left(\frac{1}{x+\sqrt{x^2+1}}\right)}}{2} \quad .$$

$$= \frac{x+\sqrt{x^2+1} - \frac{1}{x+\sqrt{x^2+1}}}{2} \quad \cdot \quad \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}$$

$$= \frac{x^2+2x\sqrt{x^2+1}+x^2+1-1}{2(x+\sqrt{x^2+1})}$$

$$= \frac{2x^2+2x\sqrt{x^2+1}}{2(x+\sqrt{x^2+1})} = \frac{\cancel{2}x(\cancel{x+\sqrt{x^2+1}})}{\cancel{2}(\cancel{x+\sqrt{x^2+1}})} = x \checkmark .$$

$$g(f(x)) = \ln\left(\frac{e^x-e^{-x}}{2} + \sqrt{\left(\frac{e^x-e^{-x}}{2}\right)^2+1}\right)$$

$$= \ln\left(\frac{e^x-e^{-x}}{2} + \sqrt{\frac{e^{2x} - 2 + e^{-2x}}{4} + 1}\right)$$

$$-\frac{2}{4} + \frac{4}{4} = +\frac{2}{4}$$

$$= \ln\left(\frac{e^x-e^{-x}}{2} + \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}}\right)$$

$$= \ln\left(\frac{e^x-e^{-x}}{2} + \sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2}\right)$$

$$= \ln\left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)$$

$$= \ln\left(\frac{2e^x}{2}\right) = \ln(e^x) = x.$$

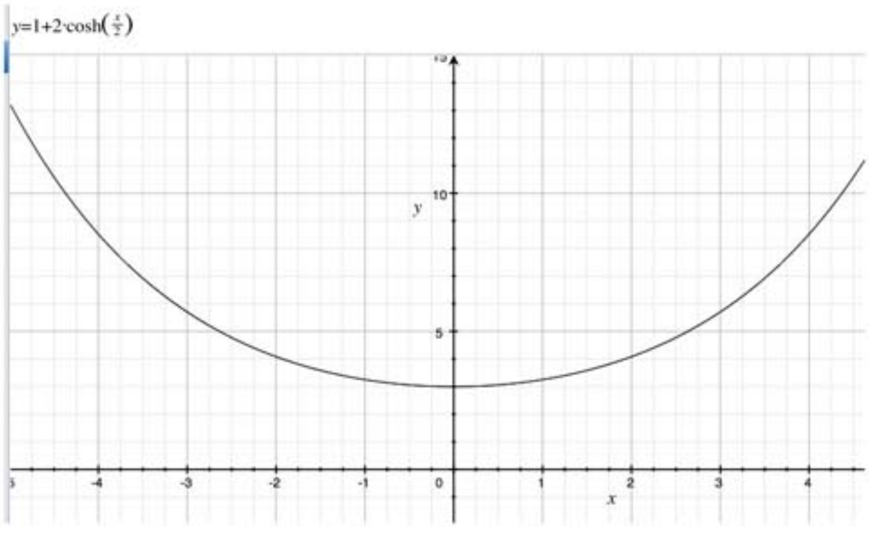
Catenary curves:

A flexible cable always hangs in the shape of a catenary curve

$$y = c + a\cosh\left(\frac{x}{a}\right) \qquad c, a \text{ constants}$$

$$a > 0$$

Ex



$$y = 1 + 2\cosh\left(\frac{x}{2}\right)$$

many applications, including architecture:

St. Louis Gateway Arch,

works of Antoni Gaudi ← google:
antoni gaudi catenary