Math 20200
Calculus II

Lesson 7
Inverse Trigonometric Functions

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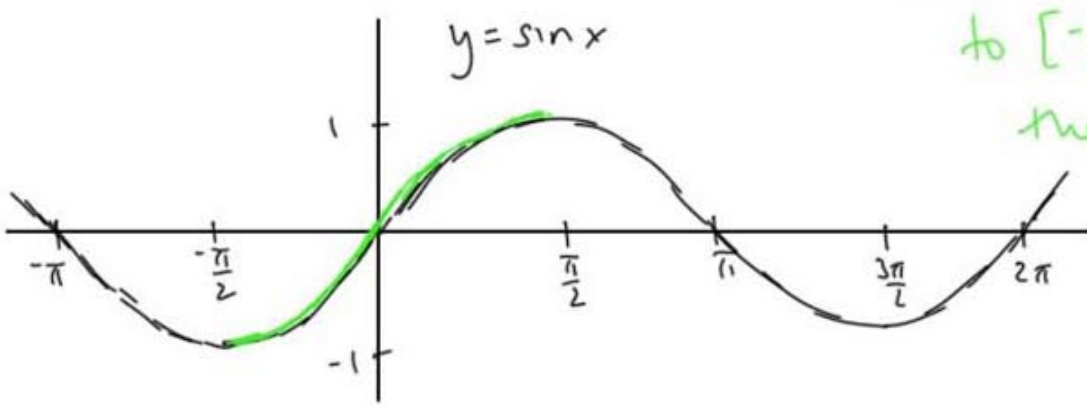
Inverse Trigonometric Functions

Recall from lesson 1 that the only functions to have inverse functions are the one to one functions.

Trig functions are not one to one on their entire domains, but we can restrict the domains, and have inverse functions.

For $f(x) = \sin x$:



restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that $y = \sin x$ is one to one.

We call the inverse function $f^{-1}(x) = \arcsin x$
$= \sin^{-1} x$.

$y = \sin x$ Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$

$y = \arcsin x$ Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$

saying $y = \arcsin x$ same as $x = \sin y$

Ex. $\sin(\frac{\pi}{2}) = 1$ and $\frac{\pi}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\therefore$ $\frac{\pi}{2} = \arcsin(1)$

Ex. $\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$ and $-\frac{\pi}{4} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\therefore$ $-\frac{\pi}{4} = \arcsin(-\frac{\sqrt{2}}{2})$

Ex. $\underbrace{\arcsin(\frac{1}{2})}_{\theta} = \frac{\pi}{6}$

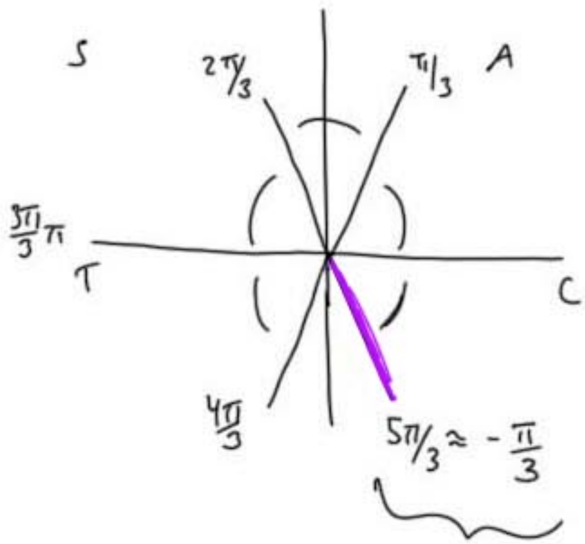
$\sin\theta = \frac{1}{2}$ $\underline{\underline{\text{and}}}$ $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ range of $\arcsin x$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.

Ex. find $\underbrace{\arcsin\left(-\frac{\sqrt{3}}{2}\right)}_{\theta} = -\frac{\pi}{3}$.

$\sin\theta = -\frac{\sqrt{3}}{2}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ range of arcsinx

reference angle $\frac{\pi}{3}$



need $\sin\theta = -\frac{\sqrt{3}}{2}$

sine negative in QIII + QIV

$\theta \in \left[-\pi/2, \frac{\pi}{2}\right]$ QI + QIV

$$\boxed{\theta = -\frac{\pi}{3}}$$

$\frac{5\pi}{3}$ + $-\frac{\pi}{3}$ are coterminal

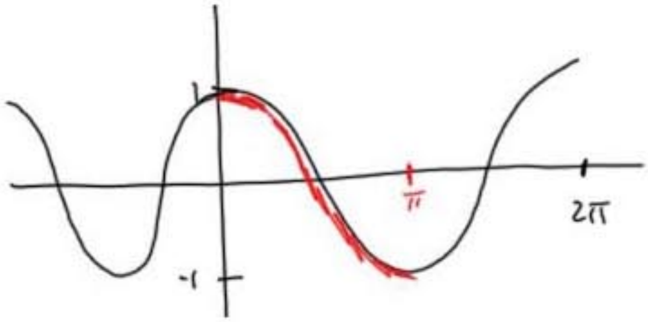
but only $-\frac{\pi}{3} \in \left[-\pi/2, \frac{\pi}{2}\right]$.

Notice, by composition of inverse functions,

$$\arcsin(\sin(x)) = x \quad \text{for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin(\arcsin(x)) = x \quad \text{for } x \in [-1, 1]$$

For $f(x) = \cos x$:



$f^{-1}(x) = \arccos x$

$= \cos^{-1} x$

$y = \cos x \quad D: [0, \pi]$

$R: [-1, 1]$

$y = \arccos x$
same as $x = \cos y$

$y = \arccos x \quad D: [-1, 1]$
$= \cos^{-1} x \quad R: [0, \pi]$

Ex. $\cos\left(\frac{\pi}{2}\right) = 0$ and $\frac{\pi}{2} \in [0, \pi]$ $\therefore$ $\frac{\pi}{2} = \arccos(0)$.

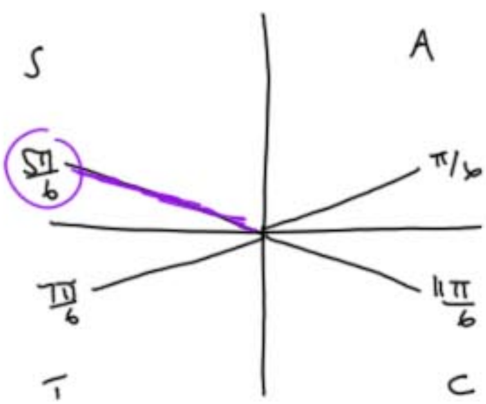
Ex. $\cos(0) = 1$ and $0 \in [0, \pi]$ $\therefore$ $0 = \arccos(1)$

Ex. $\underbrace{\arccos\left(-\frac{\sqrt{3}}{2}\right)}_{\theta} = \frac{5\pi}{6}$.

$\cos\theta = -\frac{\sqrt{3}}{2}$ and $\theta \in [0, \pi]$ range of $\arccos x$

we know $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

so $\frac{\pi}{6}$ reference angle



$\cos\theta$ negative
QII, III

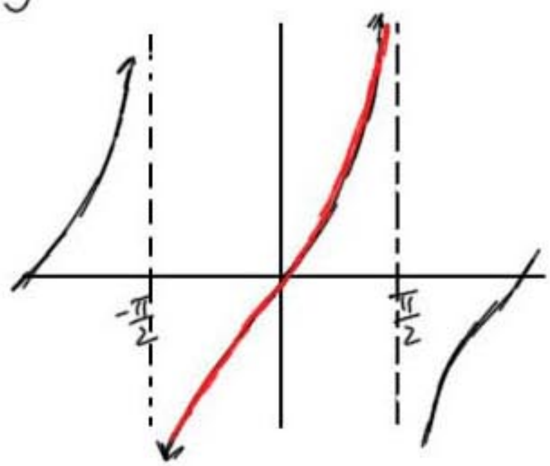
$\theta \in [0, \pi]$
QI, QII

For $f(x) = \tan x$:

$y = \arctan x$ same as $x = \tan y$

$$f^{-1}(x) = \arctan x = \tan^{-1} x$$

$y = \tan x$



$y = \tan x$ D: $(-\pi/2, \pi/2)$

R: $\mathbb{R}$ (all reals)

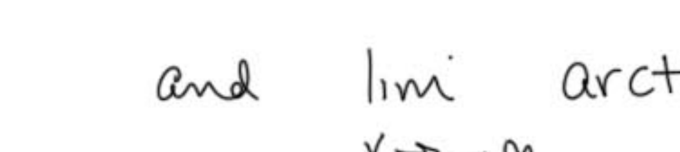
$y = \arctan x$ D: $\mathbb{R}$

R: $(-\pi/2, \pi/2)$

Ex. $\tan\left(\frac{\pi}{4}\right) = 1$ and $\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\therefore \frac{\pi}{4} = \arctan(1)$

Ex. for $y = \tan x$, as $x \to \frac{\pi}{2}^-$, $y \to \infty$

$\therefore$ for $y = \arctan x$, as $x \to \infty$, $y \to \frac{\pi}{2}^-$



$$\therefore \lim_{x \to \infty} \arctan x = \frac{\pi}{2}$$

$$\text{and} \quad \lim_{x \to -\infty} \arctan(\underbrace{e^{-x}}_{\infty}) = \frac{\pi}{2}$$

The other inverse trig functions :

for $f(x) = \csc x$ and $x \in (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

$$f^{-1}(x) = \operatorname{arccsc} x = \csc^{-1} x$$

for $f(x) = \sec x$ and $x = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$$f^{-1}(x) = \operatorname{arcsec} x = \sec^{-1} x$$

for $f(x) = \cot x$ and $x \in (0, \pi)$

$$f^{-1}(x) = \operatorname{arccot} x = \cot^{-1} x$$

Derivatives and Integrals with Inverse Trig Functions :

To find $\frac{d}{dx}(\arcsin x)$, use the relationship between the derivatives of inverse functions :

$f(x) = \sin x$

$g(x) = \arcsin x$

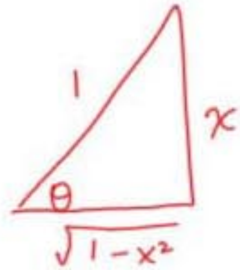
$f'(x) = \cos x$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\underbrace{\cos(\arcsin x)}_{\text{simplify}}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$\cos(\underbrace{\arcsin x}_{\theta})$

$\sin\theta = x = \frac{x}{1} \quad \frac{opp}{hyp}$



$b^2 + x^2 = 1^2$

$b^2 = 1 - x^2$

$b = \sqrt{1-x^2}$

need $\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\therefore \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.

Can also show this using implicit differentiation: $y = \arcsin x$

$\Rightarrow x = \sin y$

$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$

$1 = \cos y \cdot \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$ need in terms of x

$\cos y = \dfrac{\sqrt{1-x^2}}{1}$

$= \dfrac{1}{\sqrt{1-x^2}}$.

Then $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

note, $\arcsin x$ only exists for $-1 \le x \le 1$,
and $\dfrac{1}{\sqrt{1-x^2}}$ exists only for $-1 < x < 1$.

(for the integration rule to hold, there can't be any x-values for which $\frac{1}{\sqrt{1-x^2}}$ exists, for which $\arcsin x$ does not exist.)

Ex. find $\frac{d}{dx}(\arctan x)$

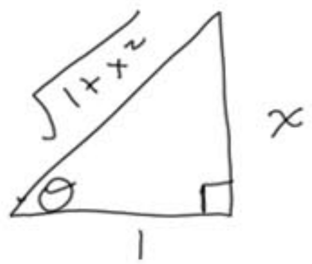
Work on this problem
on your own

$f(x) = \tan x \qquad f'(x) = \sec^2 x$

$g(x) = \arctan x$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sec^2(\arctan x)}$$

Let $\arctan x = \theta$, then $\tan\theta = \frac{x}{1} = \frac{\text{opp}}{\text{adj}}$



$$\sec^2\theta = (\sec\theta)^2$$

$$= \left(\frac{\text{hyp}}{\text{adj}}\right)^2$$

$$= \left(\frac{\sqrt{1+x^2}}{1}\right)^2 = 1+x^2$$

$$\therefore g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sec^2(\arctan x)} =$$

$$= \frac{1}{\sec^2\theta} = \frac{1}{1+x^2}.$$

$$\therefore \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

and so $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C.$

Inverse Trig Derivatives

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{x\sqrt{x^2-1}}$$

Ex. $f(x) = \arctan\left(\frac{1}{x}\right)$. find $f'(x)$

chain rule: $\displaystyle\frac{d}{dx}\Big(\arctan(g(x))\Big) = \frac{1}{1+(g(x))^2}\cdot g'(x)$

$$\therefore\ f'(x) = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)$$

$$= \frac{-1}{\left(1+\frac{1}{x^2}\right)\left(x^2\right)} = \frac{-1}{x^2+1} = \frac{d}{dx}\left(\operatorname{arccot}(x)\right).$$

notice

So, is $\arctan(\frac{1}{x}) = \operatorname{arccot} x \quad \forall x \neq 0$?

let $\theta = \arctan(\frac{1}{x})$

$$\tan\theta = \frac{1}{x} \Rightarrow \cot\theta = \frac{x}{1} = x. \quad \checkmark$$

Ex. $\displaystyle\int_0^{\frac{1}{8}} \frac{1}{\sqrt{1-16x^2}}\,dx$ similar to $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \int_0^{1/8} \frac{1}{\sqrt{1-(4x)^2}}\,dx$$

let $u = 4x$, $du = 4\,dx$ $x=0,\ u=0$; $x=\frac{1}{8},\ u=\frac{1}{2}$

$$= \frac{1}{4}\int_0^{1/8} \frac{4}{\sqrt{1-(4x)^2}}\,dx = \frac{1}{4}\int_0^{1/2}\frac{1}{\sqrt{1-u^2}}\,du =$$

$$= \frac{1}{4}\arcsin(u)\Big|_0^{1/2} = \frac{1}{4}\left(\arcsin(\tfrac{1}{2}) - \arcsin(0)\right) =$$

$$= \frac{1}{4}\left(\frac{\pi}{6} - 0\right) = \boxed{\frac{\pi}{24}}$$