Math 20200
Calculus II

Lesson 6
Exponential Growth and Decay

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Exponential Growth + Decay

Def. A **differential equation** $(\text{Diff } E_q)$ is an equation involving a function (y) and its derivative(s).

Ex. $y'' - 2y' + y = \sin x$

$$xy' - y^2 = e^{2x} + 3$$

A **solution to a diff. eq.** is a function $y = f(x)$ that makes the diff. eq. true.
(satisfies the diff. eq.)

Ex. $y' = y$.

$y = e^x$ is a solution to the diff. eq. $y' = y$

$$\text{because } y' = e^x$$
$$= y \checkmark$$

but, $y = 2e^x$ is also a solution:

$$y' = 2e^x = y \checkmark$$

$y = Ce^x$ is <u>the general solution</u> to the diff. eq. $y' = y$.

(a family of solutions)

proof: $y' = Ce^x = y. \checkmark$

What about $y = e^{x+2}$ is This also a solution?

$$y' = e^{x+2}(1) = y \checkmark$$

but $y = e^{x+2} = e^x e^2 = \overbrace{e^2}^{C} e^x$

so This is part of The $y = Ce^x$ family.

Ex. $y' = ky$ k constant $\neq 0$.

Ex. $y' = 5y$

Work on this problem
on your own

for $y' = ky$

$y = e^{kx}$ is a solution

$y' = ke^{kx} = ky$ ✓.

Ex. $y' = 5y$

$y = e^{5x}$ is
a solution.

$y' = 5e^{5x} = 5y$ ✓

and $y = Ce^{kx}$ is the general solution to
the diff. eq. $y' = ky$.

check it is a solution

$$y' = C \cdot ke^{kx} = k\underbrace{Ce^{kx}}_{y} = ky. \checkmark$$

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The differential equation $y' = ky$ is

a common model for real world problems.

(This is the only diff. eq. we'll use this lesson.)

$$y' = \underline{k} \cdot y$$

↑
rate of change   is proportional to   current
of amount   amount

population growth
    rate of change of population is
    proportional to current population

decay of radioactive elements
    rate of change of amount of element
    is proportional to the current amount
                              present,

then since $y' = ky$, we know

$k =$ relative
  growth/decay
  rate.

$$y = Ce^{kx}$$

or $\quad y = Ce^{kt} \qquad$ t time

Ex. A cell of the bacteria E. coli divides into
2 cells every 20 minutes. A culture of
this bacteria starts with 60 cells.

bacteria population, use P instead of y.

$$P' = kP \qquad \text{and} \quad P = Ce^{kt}$$

notice that at $t = 0$, $\quad P = Ce^{k(0)} = \underset{P_0}{\underline{C}}$

initial population

$$P = P_0 e^{kt} \qquad \text{given } P_0 = 60$$

$$P = 60e^{kt}$$

We also know that each cell divides into two cells every 20 minutes.
$$= \tfrac{1}{3} \, hr. \quad \leftarrow \text{parts b-d below ask in terms of hours}$$

$$120 = 60 e^{k\left(\frac{1}{3}\right)}$$

solve for k
$$2 = e^{\frac{1}{3} k}$$

$$\ln 2 = \tfrac{1}{3} k$$

$$3 \ln 2 = k$$

$$\ln 2^3 = k$$

$$\ln 8 = k. \quad \text{relative growth rate}.$$

So
$$P = 60 \, e^{(\ln 8) t} = 60 \left(e^{\ln 8}\right)^t$$
$$= 60 \cdot 8^t$$

a) find the relative growth rate (ie, k)

$$k = \ln 8, \text{ above}.$$

b) find an expression for the number of

cells after t hours.

$$P = 60 \cdot 8^t \text{, above.}$$

c) find the number of cells after 6 hours

$$P = 60 \cdot 8^6 = 15,728,640$$

leave in
this form on exam

d) find the rate of growth after 6 hours

$$P'$$

we know $P' = kP = (\ln 8) \, 60 \cdot 8^6 \quad \dfrac{\text{cells}}{\text{hr}}$

e) when will the population reach 20,000 cells?

ie, find t $\Rightarrow$ $P(t) = 20,000$

$$P = 60 \cdot 8^t$$

$$20{,}000 = 60 \cdot 8^t$$

$$\frac{20{,}000}{60} = \frac{1000}{3} = 8^t$$

$$\ln\left(\frac{1000}{3}\right) = t \ln 8$$

$$t = \frac{\ln\left(\frac{1000}{3}\right)}{\ln 8} \text{ hours.} \quad \approx 2.8 \text{ hours}$$
$$\approx 2 \text{ hrs } 48 \text{ min}$$

Note if $k > 0$, exponential growth function

k = relative growth rate

if $k < 0$, exponential decay function

$-k$ = relative decay rate

def. <u>half life</u> = time it takes for an amount to be reduced by $\frac{1}{2}$

<u>doubling time</u> = time it takes for an amount to double.

Ex. You have an 800 mg sample of radioactive element X, which has a half-life of 5 days.

a) Find a formula for mass remaining after t days.

b) Find the mass remaining after 30 days.

c) When is the mass reduced to 1 mg?

d) Sketch the graph of the mass function you found in part a)

STOP Work on this problem on your own

a) $y = y_o e^{kt}$ Given $y_o = 800$

↖ initial amount

$$y = 800 \, e^{kt}$$

$$400 = 800 \, e^{k \cdot 5} \qquad \text{half life} = 5 \text{ days}$$

$$\frac{400}{800} = e^{5k}$$

$$\frac{1}{2} = e^{5k}$$

$$\ln \tfrac{1}{2} = 5k$$

$$k = \frac{\ln \tfrac{1}{2}}{5} \qquad \Rightarrow \qquad y = 800 \left(e^{\left(\ln \tfrac{1}{2} \right) \tfrac{1}{5} t} \right)$$

$$y = 800 \left(\tfrac{1}{2} \right)^{t/5}.$$

b) $y = 800 \left(\tfrac{1}{2} \right)^{30/5} = 800 \left(\tfrac{1}{2} \right)^{6} = \dfrac{\cancel{800}^{\,100}}{\underset{8}{\cancel{64}}} = \dfrac{25}{2}$

$$= 12.5 \text{ mg}.$$

c) $1 = 800 \left(\tfrac{1}{2} \right)^{t/5}$

$$\frac{1}{800} = \left(\tfrac{1}{2} \right)^{t/5}$$

$$\ln\left(\tfrac{1}{800}\right) = \tfrac{t}{5}\ln\left(\tfrac{1}{2}\right)$$

$$t = \frac{5\ln\left(\tfrac{1}{800}\right)}{\ln\left(\tfrac{1}{2}\right)} = \frac{-5\ln\left((800)^{-1}\right)}{-\ln\left((2)^{-1}\right)} = \frac{5\ln 800}{\ln 2}$$

$$\approx 48 \text{ days}.$$

d)

remember:

$y = \left(\tfrac{1}{2}\right)^x$

$y = 2^x$

we have

$$y = 800\left(\tfrac{1}{2}\right)^{t/5}$$

our graph $\quad y = 800\left(\tfrac{1}{2}\right)^{t/5}$

$$= 800\left(\left(\tfrac{1}{2}\right)^{1/5}\right)^t$$

$t \geq 0$

$(0, 800)$

$(30, 12.5)$

$(48, 1)$

800

30   48

Newton's Law of Cooling

let T = temperature of object

T_s = surrounding (air) temperature

t = time

$$\frac{dT}{dt} = k\,(T - T_s)$$

T_s constant

$\underbrace{\frac{dT}{dt}}_{y'}$ = $k\,(\underbrace{T - T_s}_{y})$

$$y' = \frac{d}{dt}(T - T_s)$$

$$= \frac{dT}{dt} - 0$$

then $y = Ce^{kt}$

$$T - T_s = Ce^{kt}$$

$$T = Ce^{kt} + T_s$$

notice initial temperature

$T_0 = C + T_s$

Ex. A thermometer is taken from a warm room (20°C) to a cold room (5°C). After one minute, The Thermometer reads 12°C.

Initial temp of Therm 20°C.

$$T(0) = 20 = C + T_s$$

given $T_s = 5$ (cold room 5°C)

$t = 1$, $T(1) = 12$

$$T = Ce^{kt} + T_s$$

given $T_0 = 20$ and $T_s = 5$

so $20 = Ce^{k(0)} + 5$

$$20 = C + 5$$

$$C = 15$$

$$T = 15e^{kt} + 5 \qquad \text{now need } k$$

given at $t = 1$, $T = 12$

$$12 = 15e^{k(1)} + 5$$

$$7 = 15e^k$$

$$\frac{7}{15} = e^k \qquad k = \ln\left(\frac{7}{15}\right)$$

$$T = 15\,e^{\ln\left(\frac{7}{15}\right)t} + 5 \qquad\qquad e^{\ln\left(\frac{7}{15}\right)} = \frac{7}{15}$$

$$= 15\left(e^{\ln\frac{7}{15}}\right)^t + 5$$

$$T = 15\left(\frac{7}{15}\right)^t + 5$$

a) What temperature will the thermometer read after one more minute?

after one more minute, at $t = 2$

$$T = 15\left(\frac{7}{15}\right)^2 + 5$$

$$= 15 \cdot \frac{49}{15 \cdot 15} + 5 = 3\frac{4}{15} + 5 = 8\frac{4}{15}\,°C$$
$$\approx 8°C$$

b) when will the thermometer read $6°C$?

find $t \Rightarrow T = 6°$

$$6 = 15\left(\frac{7}{15}\right)^t + 5$$

$$1 = 15\left(\frac{7}{15}\right)^t$$

$$\frac{1}{15} = \left(\frac{7}{15}\right)^t$$

$$\ln\left(\frac{1}{15}\right) = t \ln\left(\frac{7}{15}\right)$$

$$t = \frac{\ln\left(\frac{1}{15}\right)}{\ln\left(\frac{7}{15}\right)} \approx 3.6 \text{ minutes}$$