Math 20200
Calculus II

Lesson 5
The Number *e* as a limit

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The Number e as a limit

We know that for $f(x) = \ln x$, $f'(x) = \frac{1}{x}$. Then

$$1 = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x \to 0} \frac{\ln(1+x)}{x} =$$

$$= \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \ln\left((1+x)^{1/x}\right)$$

$$\therefore \quad 1 = \lim_{x \to 0} \ln\left((1+x)^{1/x}\right)$$

and $$e^1 = e^{\lim_{x \to 0} \ln\left((1+x)^{1/x}\right)}$$

by continuity of $y = e^x$ $$= \lim_{x \to 0} e^{\ln\left((1+x)^{1/x}\right)} = \lim_{x \to 0} (1+x)^{1/x}$$

$$\therefore \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$

If we let $n = \frac{1}{x}$ for $x > 0$, This becomes:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

✳ base changing at same time as exponent changing

Notice, in both of These limits:

as $\underline{n \to \infty}$ $\left(\underbrace{1 + \cancel{\tfrac{1}{n}}^{\,0}}_{\to 1}\right)^{n \to \infty}$

but This is $\underline{\underline{\text{not}}}$ 1^∞ because The base never $= 1$.

approaches 1 as The exponent $\to \infty$.

Now, consider $\displaystyle\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$

base $\to 1$
exp $\to \infty$
think e.

change of variables let $m = \frac{n}{x}$

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n \to \infty} \left(1 + \frac{1}{m}\right)^{mx}$$

as $n \to \infty$
$m \to \infty$

$$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx}$$

$$= \lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^{m}\right)^{x}$$

by continuity of a^x exponential function

$$= \Big(\underbrace{\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}}_{e}\Big)^{x}$$

$$= e^x$$

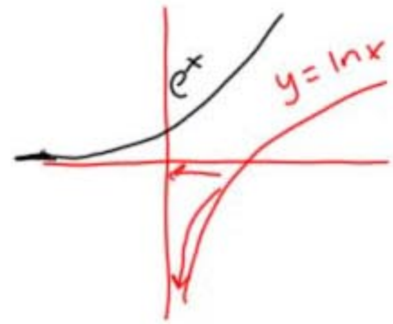
Another limit with the base + exponent changing:

Ex. $\lim_{x\to 0^+} x^{1/x}$ (Using Logs to find limits)

let $y = \lim_{x\to 0^+} x^{1/x}$

$$\ln y = \ln\left(\lim_{x\to 0^+} x^{1/x}\right)$$

pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak by continuity of $y = \ln x$ $\quad = \lim_{x \to 0^+} \ln\left(x^{1/x}\right)$



$$= \lim_{x \to 0^+} \underbrace{\frac{1}{x}}_{(+\infty)} \underbrace{\ln(x)}_{(-\infty)} = -\infty$$

$$\therefore \quad \ln y = -\infty$$

$$\underbrace{e^{-\infty}}_{0} = y \qquad\qquad y = 0.$$

$$\therefore \quad \lim_{x \to 0^+} x^{1/x} = 0 .$$