Math 20200
Calculus II

Lesson 4
General Logarithmic and Exponential Functions

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General Logarithmic and Exponential Functions

Here we define a^x for all real x, and its inverse function $\log_a x$. $(a>0, a\neq 1)$

So far we know how to compute a^r for rational numbers (Ex. $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$)

But we don't know how to make sense of a^x for irrational x-values (Ex, $8^{\pi} = ?$)

We only know how to compute e^x for any real x, because $e^x = \exp(x)$

$\nwarrow$ inverse of

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

So we define $a^x = e^{x \ln a}$.

Then, for example, $8^\pi = e^{\pi \ln 8}$ ie, the

number for which $\ln(\ ) = \pi \ln 8$.

Notice, this makes sense because

$$e^{x \ln a} = e^{(\ln a)x} = (e^{\ln a})^x = a^x.$$

With this definition, we still have the same rules of exponents for real x, y as we did for rational x, y in precalculus:

$$a^x a^y = a^{x+y} \qquad \frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x.$$

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Derivatives with a^x:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x\ln a}\right) \underset{\text{chain rule}}{=} \underbrace{e^{x\ln a}}_{a^x} \cdot \underbrace{\frac{d}{dx}(x\ln a)}_{\ln a}$$

remember $\ln a$ is a constant

$$\therefore \quad \frac{d}{dx}(a^x) = a^x \ln a.$$

Ex. $y = 5^x \quad y' = 5^x \ln 5$

Ex. $g(x) = x^4 \cdot 4^x$

$$g'(x) = \frac{d}{dx}(x^4)\cdot 4^x + x^4 \cdot \frac{d}{dx}(4^x)$$

$$= 4x^3 \cdot 4^x + x^4 \cdot 4^x \ln 4$$

$$= 4^x x^3 \left[4 + x\ln 4\right].$$

For compositions $a^{g(x)}$, we use the chain rule:

$$\frac{d}{dx}\left(a^{g(x)}\right) = a^{g(x)} \ln a \cdot g'(x)$$

Ex. $y = 10^{\tan\theta}$

$$y' = 10^{\tan\theta} \ln 10 \cdot \frac{d}{d\theta}(\tan\theta)$$

$$= 10^{\tan\theta} \ln 10 \cdot \sec^2\theta.$$

Integrals with a^x:

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

since $\frac{d}{dx}\left(\frac{a^x}{\ln a} + c\right) = \frac{1}{\ln a}\frac{d}{dx}(a^x) + 0$

$$= \frac{1}{\ln a} \cdot a^x \ln a = a^x.$$

Ex. $\int x 3^{x^2}\,dx$ composition, so $u = x^2$

$du = \underline{2}x\,dx$

$= \frac{1}{2}\int 2x\,3^{x^2}\,dx = \frac{1}{2}\int 3^u\,du$

$= \frac{1}{2}\cdot\frac{1}{\ln 3}\,3^u + C = \frac{1}{2\ln 3}\,3^{x^2} + C.$

The Graph of $f(x) = a^x$:

Notice That for $a > 0$, $a \neq 1$, $f(x) = a^x$

for $a > 1$, $f'(x) = \underbrace{a^x}_{>0}\underbrace{\ln a}_{>0} > 0 \ \forall x$

$f(x)$ strictly increasing. recall $\ln a = \int_1^a \frac{1}{t}\,dt$

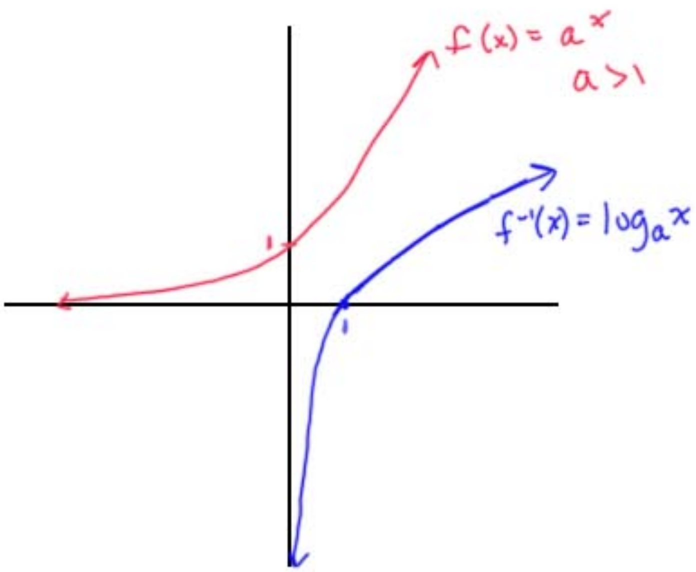
for $0 < a < 1$, $f'(x) = \underbrace{a^x}_{>0}\underbrace{\ln a}_{<0} < 0 \ \forall x$

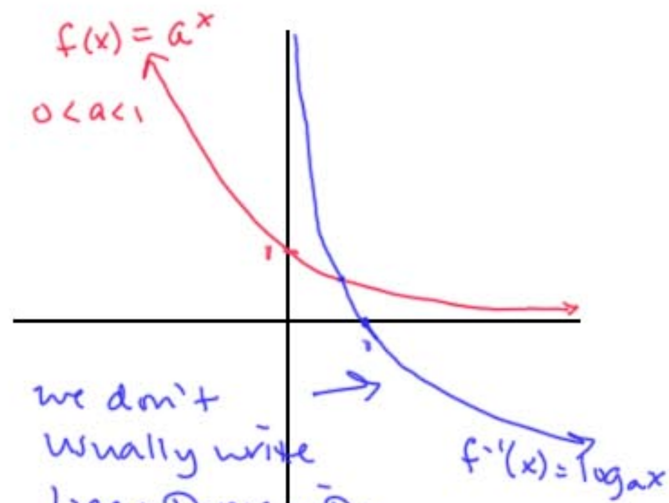
$f(x)$ strictly decreasing

$\therefore \quad f(x) = a^x$ is one to one for all $a>0,\ a\neq 1$.

also notice $f''(x) = a^x(\ln a)^2 > 0 \quad \forall x$

$\therefore$ $f(x)$ concave up for all $a>0,\ a\neq 1$.





we don't usually write logarithms with base $0<a<1$ because we can rewrite with base >1

Logarithms of base a:

Since $f(x) = a^x$ is one to one (for $a>0,\ a\neq 1$),

we can talk about its inverse, $f^{-1}(x) = \log_a x$.

ie, $\log_a x = y \iff a^y = x$

note above: $f(x) = \left(\frac{1}{3}\right)^x = (3^{-1})^x = 3^{-x} = y$

then $\log_3 y = -x$ switch $x + y$ $-y = \log_3 x$

$y = -\log_3 x$

Derivatives with log base a:

$\frac{d}{dx}(\log_a x)$ we know

$f(x) = a^x$ and $g(x) = \log_a x$

inverse functions

so $g'(x) = \dfrac{1}{f'(g(x))}$ $\qquad f'(x) = a^x \ln a$

$$= \frac{1}{\underbrace{a^{\log_a x}}_{x} \cdot \ln a} = \frac{1}{x \ln a}.$$

composition of inverse functions

$$\therefore \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}.$$

Can also find this by using implicit differentiation:

$$y = \log_a x \iff a^y = x$$

to find $\frac{dy}{dx}$: $\frac{d}{dx}(a^y) = \frac{d}{dx}(x)$

$$a^y \ln a \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}.$$

For compositions:

chain rule: $\frac{d}{dx}\left(\log_a(g(x))\right) = \frac{1}{g(x)\ln a} \cdot g'(x)$

Ex. $f(x) = \log_{10}\left(\frac{x}{x-1}\right) = \log_{10} x - \log_{10}(x-1)$

laws of logs to simplify

$$f'(x) = \frac{1}{x \ln 10} - \frac{1}{(x-1)\ln 10} \cdot \underbrace{\frac{d}{dx}(x-1)}_{1}$$

$$= \frac{1}{x\ln 10} - \frac{1}{(x-1)\ln 10}$$

Integrals with log base a:

$$\int \frac{3}{x(\log_5 x)^2}\,dx$$

notice the composition

$u = \log_5 x$

$du = \frac{1}{x\ln 5}\,dx$

$$= 3(\ln 5)\int \frac{1}{(\ln 5)x(\log_5 x)^2}\,dx = 3\ln 5 \int \frac{1}{u^2}\,du =$$

$$= 3\ln 5 \int u^{-2}\,du = 3\ln 5\,\frac{u^{-1}}{-1} + C$$

$$= \frac{-3\ln 5}{u} + C = -\frac{3\ln 5}{\log_5 x} + C$$

Logarithmic Differentiation:

look out for:

$\boxed{\begin{array}{l}\text{variable in base}\\ \text{variable in exp.}\end{array}}$

take ln( ) of both sides of equation

Ex. $y = (\sin x)^x$ find y'

$$\ln y = \ln\left((\sin x)^x\right)$$

$$\ln y = x \ln(\sin x) \quad \text{now differentiate.}$$

$$\frac{y'}{y} = \underbrace{\frac{d}{dx}(x)}_{1} \cdot \ln(\sin x) + x \cdot \frac{d}{dx}\big(\ln(\underbrace{\sin x}_{g(x)})\big)$$

$$\frac{g'(x)}{g(x)}$$

$$\frac{y'}{y} = \ln(\sin x) + x\,\frac{\cos x}{\sin x}$$

$$\frac{y'}{y} = \ln(\sin x) + x \cot x$$

$$y' = \left(\ln(\sin x) + x\cot x\right) \cdot y$$

$$y' = \left(\ln(\sin x) + x\cot x\right)(\sin x)^x.$$