# Math 20200
# Calculus II

# Lesson 03
# The Natural Exponential Function

Dr. A. Marchese, The City College of New York

Table of Contents:

# The Natural Exponential Function

We mentioned in the last lesson that in Precalculus, exponential functions were not formally defined, that $a^x$ for $x$ irrational was not presented formally. Here, we start with a definition of $e^x$ for all real $x$ (including irrationals) and in the next lesson we expand to exponential and log functions of any base $a$.

We know from Lesson 2 that $f(x) = \ln x$ is strictly increasing on its domain $(0, \infty)$.

$\therefore$ $f(x) = \ln x$ has an inverse function.

We call this inverse function $\exp(x)$, then

Show that $\exp(x) = e^x$ for rational $x$ values,

← for all real $x$

then define $e^x$ to be $= \exp(x)$.

So we define $y = \exp(x)$ to be the

inverse of $y = \ln x$ ie, $y = \exp(x) \Leftrightarrow x = \ln y$.

If $r$ is rational, laws of logs give us

$$\ln(e^r) = r\ln(e) = r(1) = r$$

we know $\ln(\exp(r)) = r$ by composition of inverse functions

since $\ln x$ is one to one, $\exp(r) = e^r$

for rational numbers $r$.

So we define $e^x = \exp(x)$ $\forall x \in \mathbb{R}$.

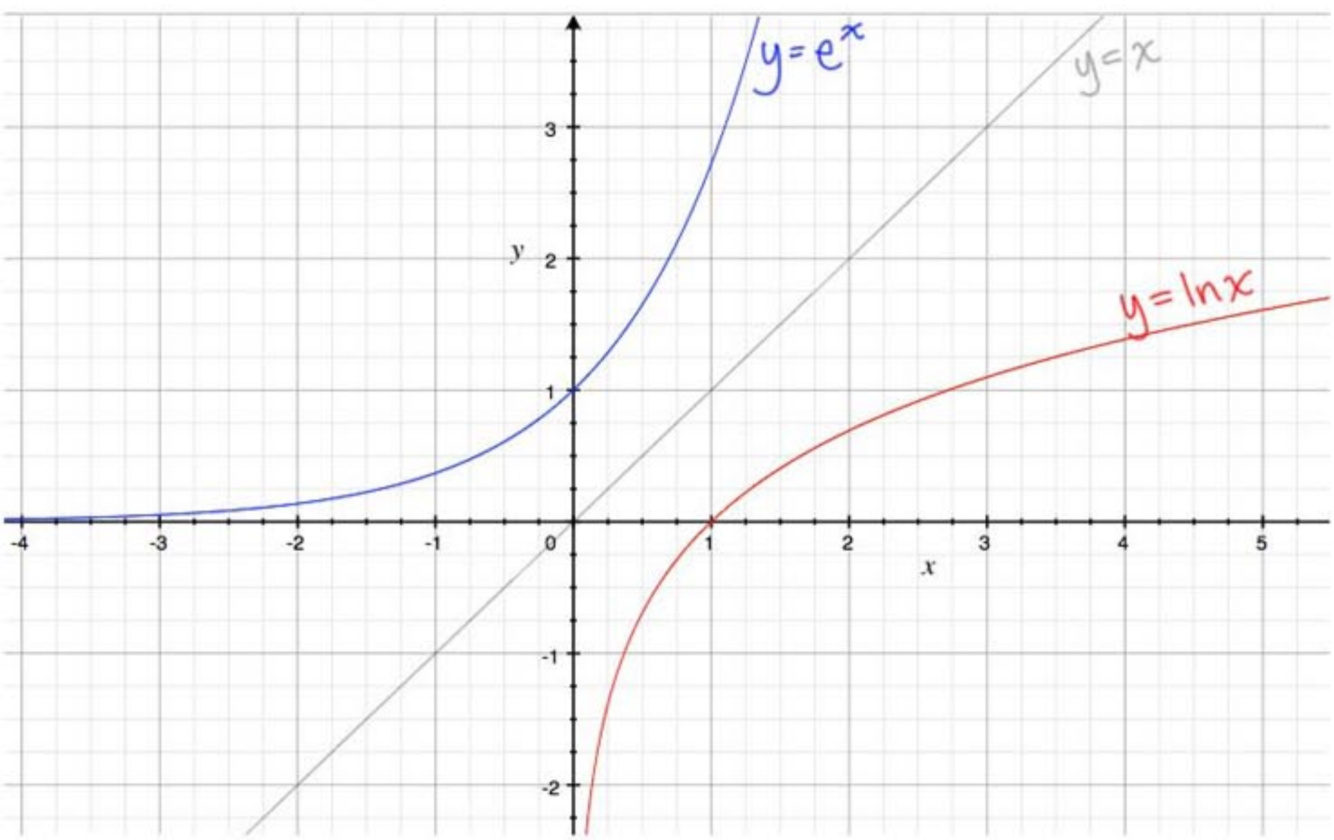
And $y = e^x$ is our <u>natural exponential function</u>.

We know:

$$y = e^x \iff x = \ln y$$

$$\left.\begin{array}{l} e^{\ln x} = x \quad \text{for } x > 0 \\ \ln(e^x) = x \end{array}\right\} \text{by composition of inverse functions}$$

$$e^0 = 1 \quad \text{since} \quad \ln(1) = 0$$



$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

Ex. $$\lim_{x \to -\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$$

$$= \lim_{x \to -\infty} \frac{e^{3x} - \frac{1}{e^{3x}}}{e^{3x} + \frac{1}{e^{3x}}} \cdot \frac{e^{3x}}{e^{3x}}$$

$$= \lim_{x \to -\infty} \frac{\overbrace{e^{\overbrace{6x}^{-\infty}}}^{\to 0} - 1}{\underbrace{e^{\overbrace{6x}^{-\infty}}}_{0} + 1} = \frac{-1}{1} = \boxed{-1.}$$

Now, we know $f(x) = \ln x$ and $f^{-1}(x) = e^x$ are inverse functions. We know $f'(x) = \frac{1}{x}$.

What is $(f^{-1})'(x)$? ie, what is $\frac{d}{dx}(e^x)$?

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(e^x)} = \frac{1}{\frac{1}{e^x}} = e^x$$

$$\therefore \frac{d}{dx}(e^x) = e^x .$$

chain rule $\quad \frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} \cdot g'(x) .$

Ex. $y = e^{5x^2}$

$$y' = e^{5x^2} \cdot \underbrace{\frac{d}{dx}(5x^2)}_{10x} = 10x\, e^{5x^2}$$

Ex. $y = e^x \ln x$

**STOP** Work on this problem on your own

$$y' = \frac{d}{dx}(e^x)\ln x + e^x \frac{d}{dx}(\ln x) \quad \text{product rule}$$

$$= e^x \ln x + e^x \cdot \frac{1}{x}$$

$$= e^x\left(\ln x + \frac{1}{x}\right).$$

Ex. $y = \cos(e^{\pi x})$ double chain


Work on this problem
on your own

$$y' = -\sin(e^{\pi x}) \cdot \underbrace{\frac{d}{dx}(e^{\pi x})}_{\text{chain rule}} \quad \text{chain rule}$$

$$= -\sin(e^{\pi x}) \cdot e^{\pi x} \cdot \underbrace{\frac{d}{dx}(\pi x)}_{\pi}$$

$$= -\pi e^{\pi x} \sin(e^{\pi x}).$$

Since $\frac{d}{dx}(e^x) = e^x$, $\int e^x\,dx = e^x + C$

Ex. $\int \frac{e^{1/x}}{x^2}\,dx$ we notice a composition of functions, inside function $\frac{1}{x}$

let $u = \frac{1}{x} = x^{-1}$

$$du = -x^{-2}\,dx$$

$$= -\frac{1}{x^2}\,dx$$

$$-\int -\frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du$$

$$= -e^u + C$$

$$= -e^{\frac{1}{x}} + C.$$

Ex. $\int_0^1 x e^{-x^2}\, dx$


Work on this problem
on your own

composition of functions
inside function is $-x^2$

so let $u = -x^2$

$du = \underline{-2}x\, dx$

$$\underset{x=0}{\overset{x=1}{\phantom{.}}} -\frac{1}{2}\int_0^1 -2x\, e^{-x^2}\, dx = -\frac{1}{2}\int_0^{-1} e^u\, du = -\frac{1}{2} e^u \Big]_0^{-1}$$

$x = 0 \quad u = -x^2, \; u = 0$

$x = 1 \quad u = -x^2, \; u = -(1)^2 = -1$

$$\left(-\tfrac{1}{2}e^{-1}\right) - \left(-\tfrac{1}{2}e^{0}\right) = \frac{-1}{2e} + \frac{1}{2}.$$

Ex. $y = \dfrac{e^x}{x}$ find an equation for the tangent line at $(1, e)$


Work on this problem on your own

slope of tangent line $= y'(1)$

$$y' = \frac{x \cdot \frac{d}{dx}(e^x) - e^x \, \frac{d}{dx}(x)}{x^2}$$

$$= \frac{xe^x - e^x(1)}{x^2} = \frac{e^x(x-1)}{x^2}$$

$$y'(1) = \frac{e^1(1-1)}{1^2} = 0 = \text{slope}$$

horizontal line. $y=$

$$y - y_1 = m(x - x_1) \quad \text{at } (1, e)$$

$$y - e = \cancel{0(x-1)} = 0 \qquad y - e = 0$$

$$\boxed{y = e}$$ horizontal line.