Math 20200
Calculus II

Lesson 2
The Natural Logarithmic Function

Dr. A. Marchese, The City College of New York

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The Natural Logarithmic Function

In Precalculus we learn about exponential functions $y = a^x$ $a>0, a\neq 1$ Ex $y=2^x$

$y=5^x$

$y=\left(\frac{1}{3}\right)^x$

and their corresponding inverse functions

$y = \log_a x$ since $y=\log_a x \iff a^y = x$

+ inverse functions switch x + y

Ex. $y=\log_2 x$ $\quad y=\log_5 x$ $\quad y=\log_{\frac{1}{3}} x$

You may have also seen $y = e^x$ for our special number $e \approx 2.718$

with inverse $y = \ln x = \log_e x$.

But in Precalculus, we can only give an intuitive idea about what it means to define a^x

when x is an irrational number. So, none of

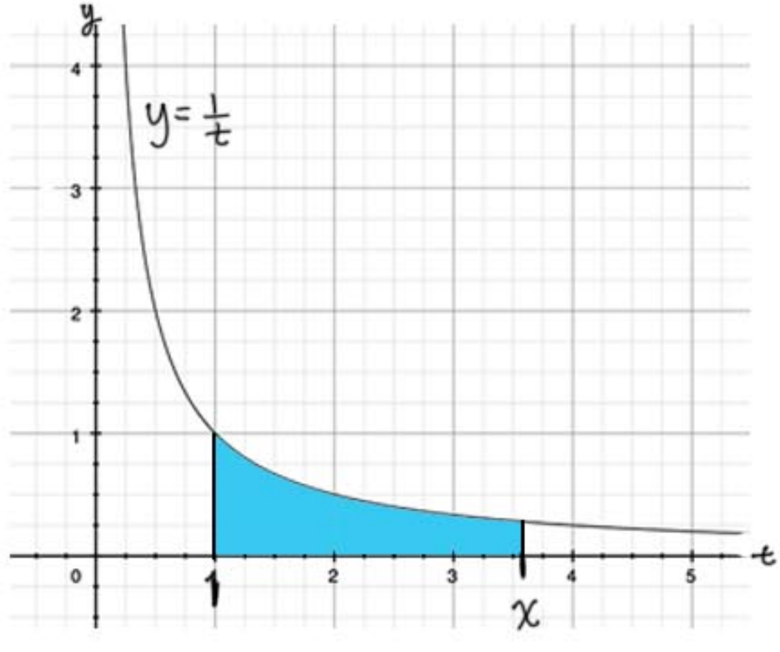
the above was defined formally in precalc.

Here, we begin a formal definition + presentation

of exponential and log functions.

We start with the <u>natural logarithmic function</u>

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad \text{for } x > 0.$$

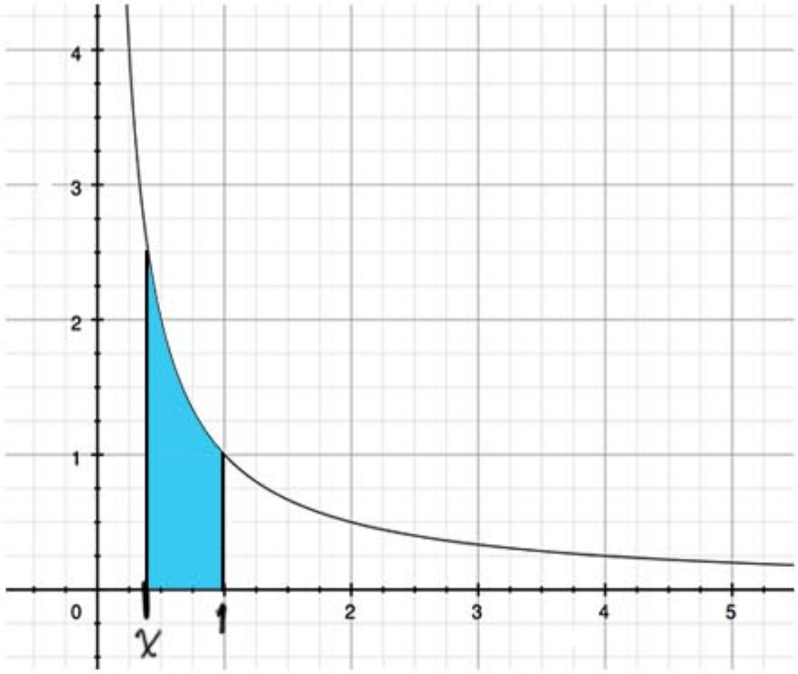


for $x > 1$

$\int_1^x \frac{1}{t}\,dt$ is the area

under $y = \frac{1}{t}$ over $[1, x]$.

for $x = 1$, $\int_1^x \frac{1}{t}\,dt = \int_1^1 \frac{1}{t}\,dt = 0 \quad \ln(1) = 0$



for $0 < x < 1$

$$\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt =$$

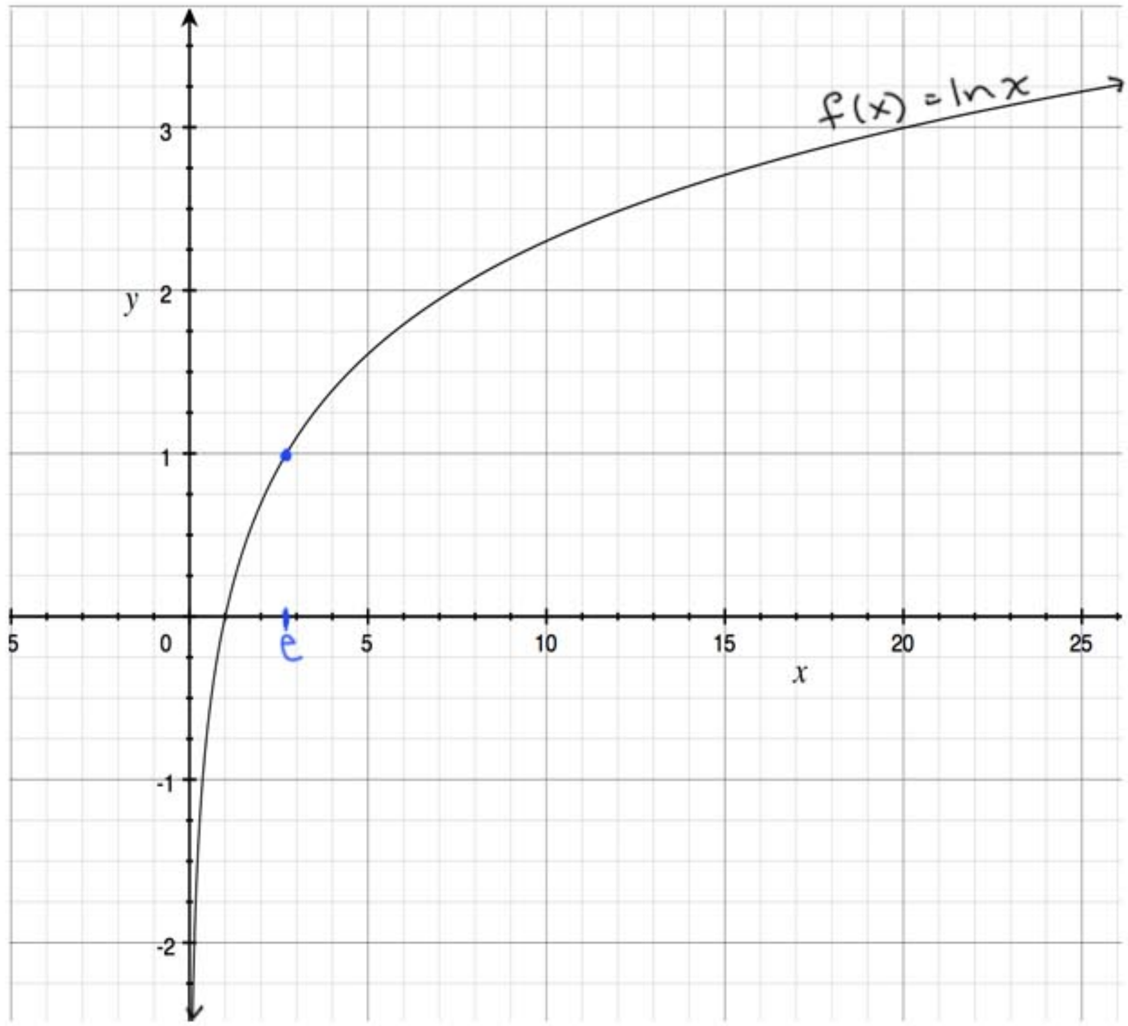
negative of the area under $y = \frac{1}{t}$ over $[x, 1]$

We also have $\lim_{x\to\infty} \ln x = \infty$, $\lim_{x\to 0^+} \ln x = -\infty$

Also note that by the Fundamental theorem of Calculus $\frac{d}{dx}\left(\ln x\right) = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}$.

Since $\ln x$ is only defined for $x > 0$, we know $\frac{d}{dx}(\ln x) = \frac{1}{x} > 0$ $\therefore$ $f(x) = \ln x$ is an increasing function and $f''(x) = -\frac{1}{x^2} < 0$ $\therefore f(x) = \ln x$ is concave down.

$\therefore$ The graph of $f(x) = \ln x$:



With this definition of $\ln x$ we have the usual laws of logarithms: for $x, y > 0$ and r rational

$\ln(xy) = \ln x + \ln y$, $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$, $\ln(x^r) = r \ln x$

And with this definition of $\ln x$, we define

e to be the number such that $\ln(e) = 1$.

(more in next lesson)

Differentiation with $\ln x$:

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}\Big(\ln\big(g(x)\big)\Big) = \frac{1}{g(x)} \cdot g'(x)$$

$$= \frac{g'(x)}{g(x)}$$

Ex. $y = \ln(x^2+10) \qquad y' = \dfrac{2x}{x^2+10}$

Ex. $y = \underbrace{x \ln x}_{\text{product rule.}} \qquad y' = \frac{d}{dx}(x) \cdot \ln x + x \cdot \frac{d}{dx}(\ln x)$

$$= 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1.$$

Ex. find $\frac{dy}{dx}$ for $\ln(xy) = y\sin x$

separate using laws of logs

$$\ln x + \ln y = y\sin x$$

implicit differentiation:

$$\frac{d}{dx}(\ln x + \ln y) = \frac{d}{dx}(\underbrace{y\sin x}_{\text{product rule}})$$

$$\frac{1}{x} + \frac{1}{y}\cdot\frac{dy}{dx} = \frac{dy}{dx}\sin x + y\cos x$$

solve for $\frac{dy}{dx}$

$$\frac{1}{y}\frac{dy}{dx} - \frac{dy}{dx}\sin x = y\cos x - \frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \sin x\right) = y\cos x - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{y\cos x - \frac{1}{x}}{\frac{1}{y} - \sin x}\cdot\frac{xy}{xy} = \frac{xy^2\cos x - y}{x - xy\sin x}.$$

Consider $y = \ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(\underbrace{-x}_{g(x)}) & x < 0 \end{cases}$

$$y' = \begin{cases} \frac{1}{x} & x > 0 \\ \frac{-1}{-x} = \frac{1}{x} & x < 0 \end{cases}$$

chain rule $\cdot \frac{g'(x)}{g(x)}$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$$

$$\therefore \int \frac{1}{x}\,dx = \ln|x| + C.$$

Ex. $\int \left(2x^2 + \frac{3}{x} - 5\right) dx =$

$3 \cdot \frac{1}{x}$

$$= \frac{2x^3}{3} + 3\ln|x| - 5x + C.$$

Ex. $\displaystyle\int \frac{2x+7}{x^2+7x-1}\,dx = \int \frac{du}{u} = \int \frac{1}{u}\,du$

look out for $\frac{g'(x)}{g(x)}$ situation

$= \ln|u| + C$

$= \ln|x^2+7x-1| + C$

let $u = g(x)$ $\quad u = x^2+7x-1$

$du = (2x+7)\,dx$

Logarithmic Differentiation

Ex. $y = \dfrac{(x^3+1)^4 \sin^2 x}{\sqrt[3]{x}}$ find y'

$\ln y = \ln\left(\dfrac{(x^3+1)^4 \sin^2 x}{\sqrt[3]{x}}\right)$

instead of two chain rules in a product in a quotient, take ln of both sides

Now expand using the laws of logarithms ...

STOP Work on this problem on your own

$$\ln y = \ln\left[(x^3+1)^4(\sin x)^2\right] - \ln\left(x^{1/3}\right)$$

$$\ln y = \ln\left((x^3+1)^4\right) + \ln\left((\sin x)^2\right) - \ln\left(x^{1/3}\right)$$

$$\ln y = 4\ln(x^3+1) + 2\ln(\sin x) - \frac{1}{3}\ln x$$

take derivative

STOP Work on this problem on your own

$$\frac{y'}{y} = 4 \cdot \frac{3x^2}{x^3+1} + 2\,\frac{\cos x}{\sin x} - \frac{1}{3}\cdot\frac{1}{x}$$

$$\frac{y'}{y} = \frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x}$$

$$y' = y\left(\frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x} \right)$$

$$y' = \frac{(x^3+1)^4 \sin^2 x}{\sqrt[3]{x}} \left(\frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x} \right).$$