Math 20200
Calculus II

Lesson 1
Inverse Fuctions

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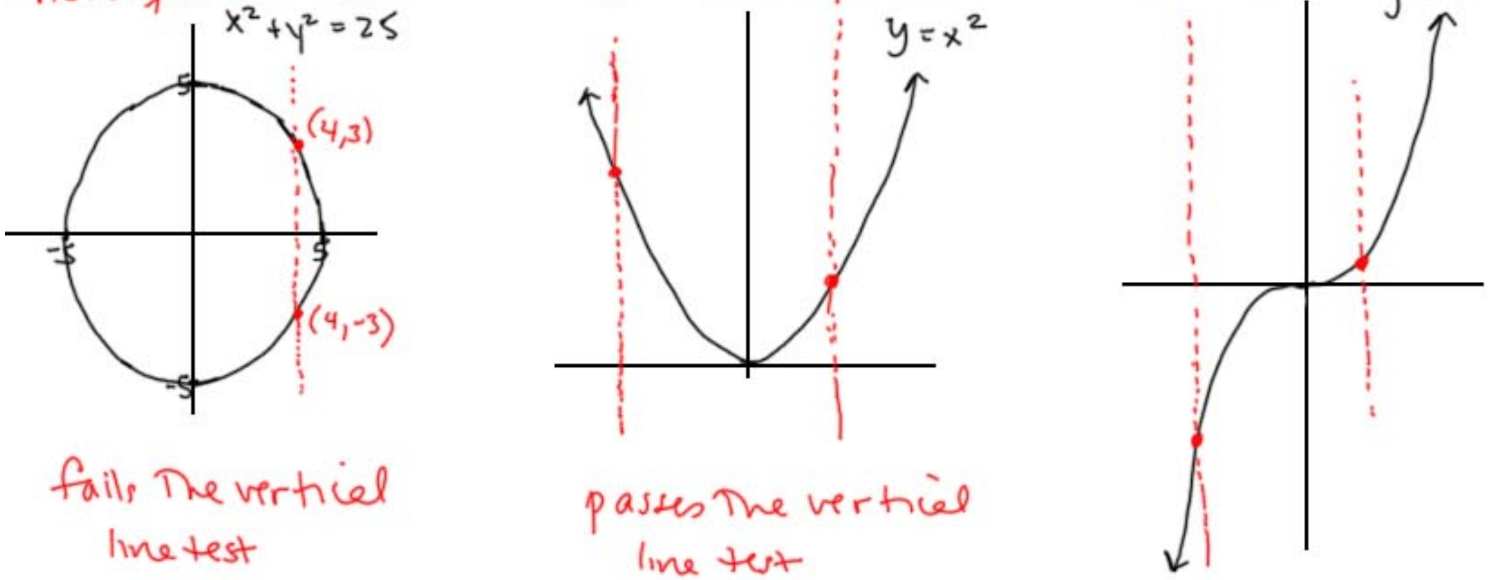
Inverse Functions

Recall: y is a <u>function</u> of x if for each x-value, there is only one corresponding y-value.

y is not a function of x | y is a function of x | y is a function of x

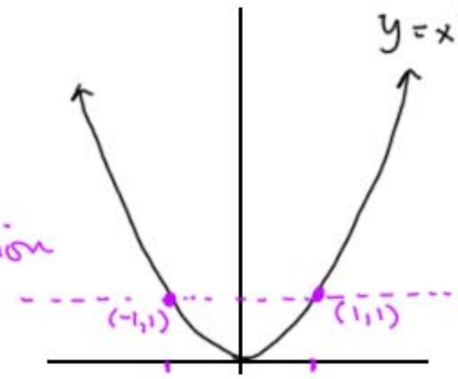


fails the vertical line test | passes the vertical line test | passes the vertical line test

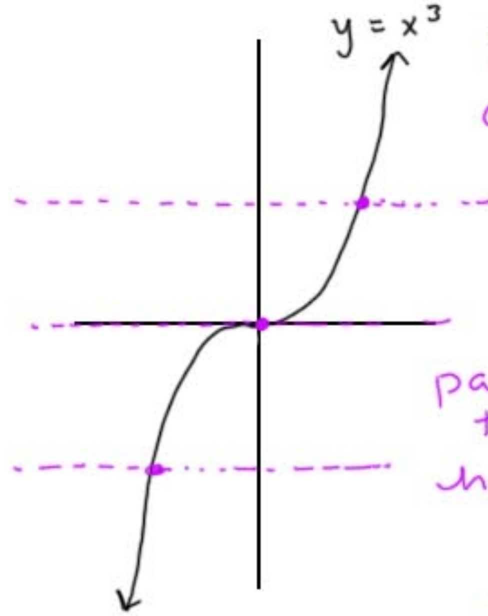
<u>vertical line test</u> to see if a graph represents a function:
any vertical line should hit the graph at most once if the graph represents a function

<u>one to one function</u> for each x-value there is only one corresponding y-value, and for each y-value, there is only one corresponding x-value.

not a one to one function



fails the horizontal line test



is a one to one function

passes the horizontal line test

<u>horizontal line test</u> to see if a function is one to one: any horizontal line should hit the graph at most once if the function is one to one

Note: for $y = x^3$ above, $y' = 3x^2 \geq 0$

$3x^2 > 0 \quad x \neq 0, \quad 3x^2 = 0 \quad x = 0$

$\therefore$ $y = x^3$ strictly increasing on $(-\infty, \infty)$

$\therefore$ $y = x^3$ is one to one.

for one to one functions, if we switch the role of x + y, we still get a function.
the inverse function

$y = x^3$ solve for x

$y^{1/3} = (x^3)^{1/3}$

$y^{1/3} = x$ switch the roles of x + y

$x^{1/3} = y$ inverse function for $y = x^3$

So for $f(x) = x^3$, $\underbrace{f^{-1}(x)}_{\text{"f inverse of x"}} = x^{1/3}$.

Properties of Inverse functions:

1) composition:

for any $f(x)$ and $f^{-1}(x)$,

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x.$$

2) one to one pairing of points

if we know $f(2) = 8$, then $f^{-1}(8) = 2$

1) composition:

Now we have $f(x)=x^3$ and $f^{-1}(x)=x^{1/3}$

Consider $f^{-1}(f(2))$

$$f^{-1}(f(2)) = f^{-1}(2^3) = f^{-1}(8) = 8^{1/3} = 2$$

the functions undo eachother.

consider $f(f^{-1}(-27))$

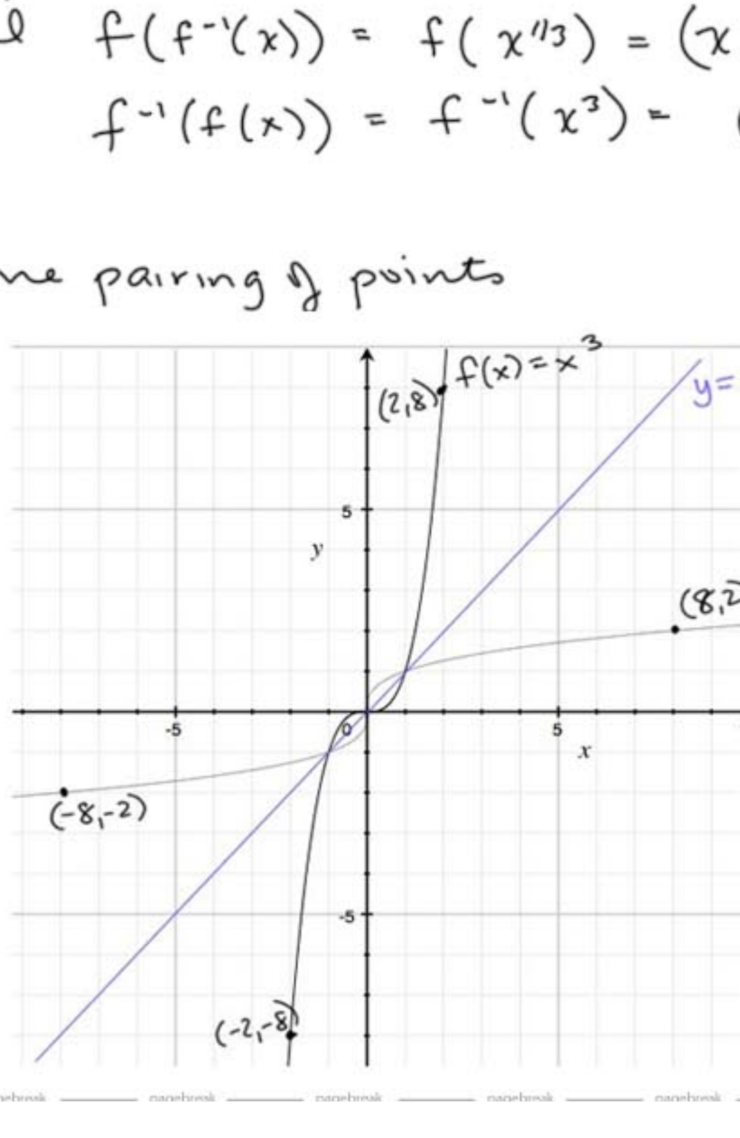
$$f(f^{-1}(-27)) = f((-27)^{1/3}) = f(-3) = (-3)^3 = -27.$$

in general (any x) $f(f^{-1}(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$

$f^{-1}(f(x)) = f^{-1}(x^3) = (x^3)^{1/3} = x.$

2) one to one pairing of points

graphs of $y=f(x)$ and $y=f^{-1}(x)$ are reflections through $y=x$



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Ex. Does $f(x) = -2x^3 - x + 7$ have an inverse?

(asking is f is one to one)

Not an easy transformations graph; to graph we'd need calculus. So instead,

check to see if $f(x)$ is strictly increasing or strictly decreasing on its domain $(-\infty,\infty)$.

$$f'(x) = -6x^2 - 1 = -(\underbrace{6x^2+1}_{>0}) < 0 \quad \text{on } (-\infty,\infty)$$

$\therefore f(x)$ is strictly decreasing on its (continuous) domain

$\therefore f(x)$ is one to one and has an inverse.

Notice, however, we can't explicitly find $f^{-1}(x)$ here.

$$y = -2x^3 - x + 7$$

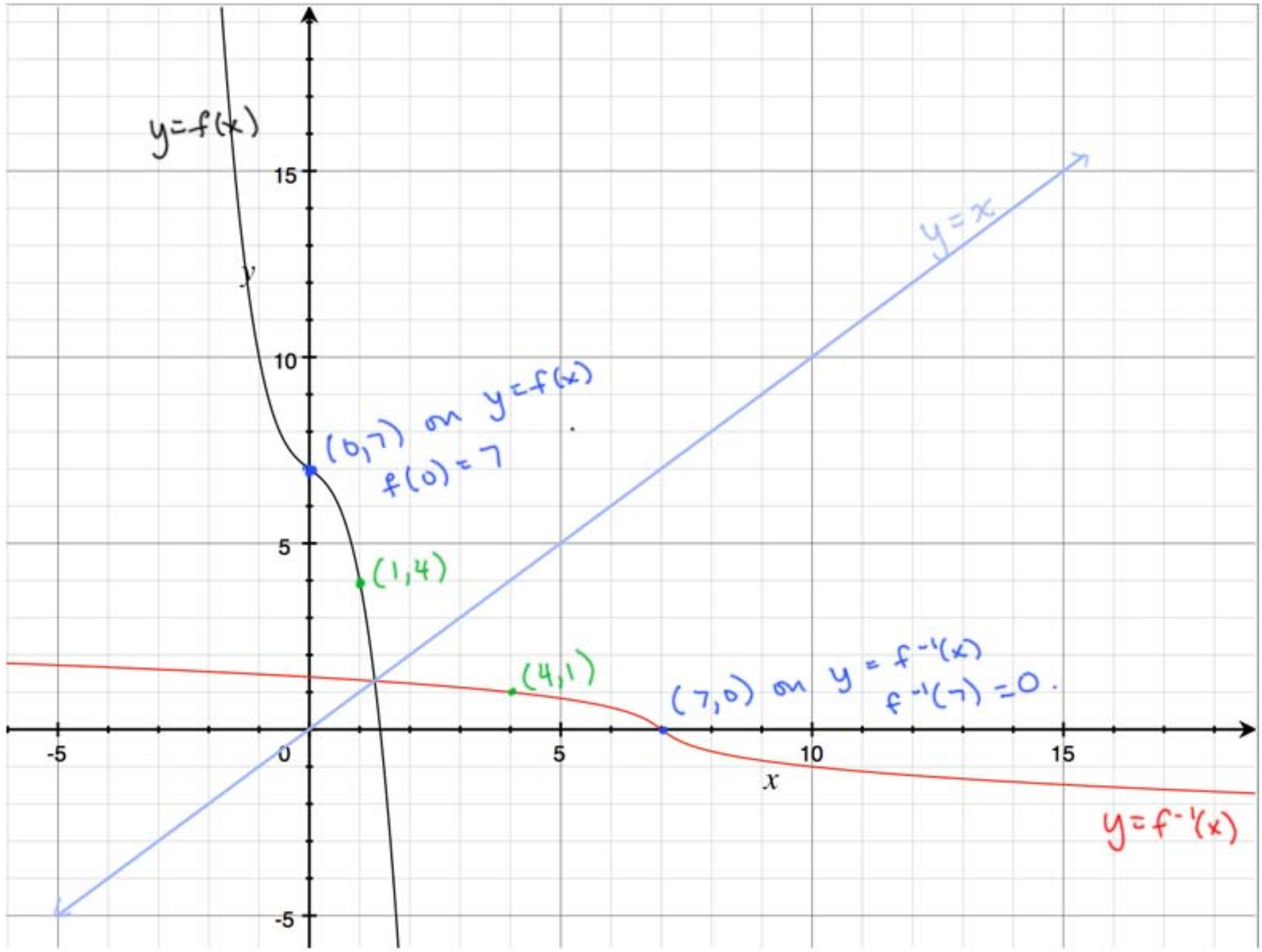
$$y - 7 = -2x^3 - x \qquad \text{we can't solve for } x.$$

We <u>can</u> find f^{-1} evaluated at particular points, though.

Ex. $f(x) = -2x^3 - x + 7 \qquad f(0) = 7 \iff f^{-1}(7) = 0.$

$$f(1) = -2 - 1 + 7 = 4$$

$$f(1) = 4 \iff f^{-1}(4) = 1$$



What if f is not one to one? Can it have an inverse? not on its entire domain, but...

If we restrict the domain of f so that f is one to one on that domain, f will have an inverse function.

Ex. $f(x) = x^2 + 4$

not one to one on $(-\infty, \infty)$

restrict to $\underline{x \geq 0}$, Then $f(x) = x^2 + 4$ is one to one

$y = x^2 + 4$ solve for x

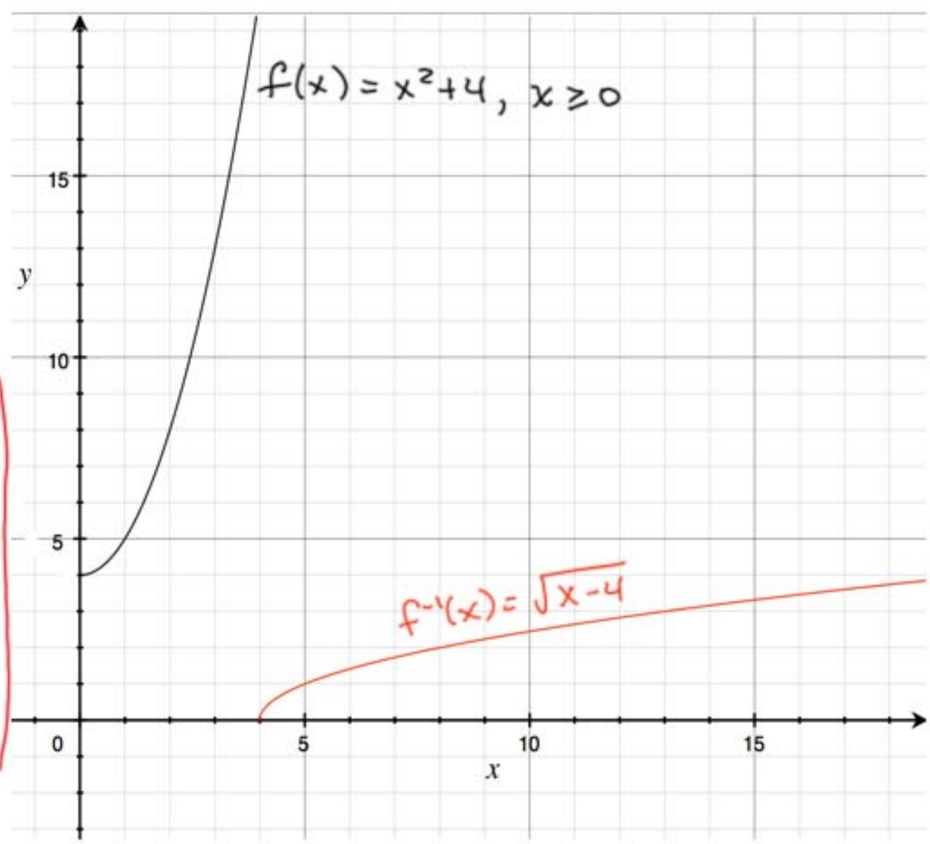
$y - 4 = x^2$

$x = + \sqrt{y-4}$ only The positive $\sqrt{\ }$ because $x \geq 0$

now switch x + y

$y = \sqrt{x-4}$

$f^{-1}(x) = \sqrt{x-4}$



for $f(x) = x^2 + 4,\ x \geq 0$

Domain: $x \geq 0$ $[0, \infty)$

Range: $y \geq 4$ $[4, \infty)$

for $f^{-1}(x) = \sqrt{x-4}$

Domain: $x \geq 4$ $[4, \infty)$

Range: $y \geq 0$ $[0, \infty)$

Notice, because we switch the roles of x + y,

Domain + Range :

domain of $f(x)$ = range of $f^{-1}(x)$

range of $f(x)$ = domain of $f^{-1}(x)$

The Calculus of Inverse Functions

Suppose $f(x)$ and $\underbrace{g(x)}_{f^{-1}(x)}$ are inverse functions

and we know $f'(x)$. We want to find $\underbrace{g'(x)}_{(f^{-1})'(x)}$;

We know that $f(g(x)) = x$

differentiate both sides $\underbrace{f'(g(x))\cdot g'(x)}_{\text{chain rule}} = 1$

$$g'(x) = \frac{1}{f'(g(x))}$$

Show that this relationship holds for

$f(x) = x^3$ and $g(x) = x^{1/3}$ inverses

we know $f'(x) = 3x^2$. (pretend we don't know $g'(x)$).

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3(x^{1/3})^2} = \frac{1}{3x^{2/3}}$$

$g'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$ ✓ same.

<u>Theorem</u>: If f is a one to one differentiable function with inverse f^{-1} and $f'(f^{-1}(a)) \neq 0$, then f^{-1} is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Proof uses the definition of the derivative.

Ex. $f(x) = x^5 - x^3 + 2x$ find $(f^{-1})'(2)$.

(We can assume f^{-1} exists and is differentiable at 2.
Or, show $f(x)$ is strictly inc or dec $\Rightarrow$ one to one
then Theorem above gives existence of $(f^{-1})'(2)$.)

To find $f^{-1}(x)$, we'd have to solve $y = x^5 - x^3 + 2x$ for x.

Can't solve for x, but notice we're not asked for

$f^{-1}(x)$. only $(\underbrace{f^{-1}}_{g})'(2)$.

$$g'(x) = \frac{1}{f'(g(x))}$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad (f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$$

What is $f^{-1}(2)$?

$f^{-1}(2)$ is the x-value when $y=2$ on $y=f(x)$

Remember, if (a,b) is on $y=f(x)$, $f(a)=b$

(b,a) is on $y=f^{-1}(x)$, $f^{-1}(b)=a$

so $f^{-1}(2)=a$ such that $f(a)=2$

or $f^{-1}(2)=x$ such that $f(x)=2$

$$f(x)=x^5-x^3+2x$$

$$x^5-x^3+2x=2$$

usually $x=0$, $x=\pm 1$.

try $x=1$ $\quad 1-1+2 \overset{\checkmark}{=} 2$

so $f(1)=2$ and $1=f^{-1}(2)$

$$(f^{-1})'(2)=\frac{1}{f'(\underbrace{f^{-1}(2)}_{1})}=\frac{1}{f'(1)}=\boxed{\frac{1}{4}}$$

$f(x)=x^5-x^3+2x$

$f'(x)=5x^4-3x^2+2 \quad f'(1)=5-3+2=4$