Math 20100
Calculus I

Lesson 29
Integration by Substitution

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Integration by Substitution

So far we have the following basic integration rules:

$$\int k \, dx = kx + C$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1 \quad (n = -1 \text{ in Calc II})$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

(also $\int \csc^2 x = -\cot x + C$

$\int \csc x \cot x = -\csc x + C$)

But what if our integrand doesn't fit the rules above?

function inside the integral

Look for a composition of functions and use u-substitution (undoes chain rule)

Ex. $\int x(4+x^2)^{10}\,dx$

① let u = inside function. we have $4+x^2$ inside $(\ )^{10}$, so $u = 4+x^2$

② compute $du = u'(x)\,dx$ (differential)

$$u = 4 + x^2$$
$$u' = 2x \Rightarrow du = 2x\,dx$$

③ rewrite the integral in terms of u and du:

$$\frac{1}{2}\int \underbrace{2x\,(\underbrace{4+x^2}_{u})^{10}\,dx}_{du} = \frac{1}{2}\int u^{10}\,du$$

④ integrate!! $$= \frac{1}{2}\cdot\frac{u^{11}}{11} + C$$

⑤ rewrite in terms of x and simplify $$= \frac{1}{22}(4+x^2)^{11} + C.$$

(⑥ check by taking the derivative)

$$\frac{d}{dx}\left(\frac{1}{22}(4+x^2)^{11} + C\right) = \frac{11}{22}(4+x^2)^{10}(2x) + 0$$
$$= x(4+x^2)^{10} \quad \checkmark$$

Ex. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

① let $u = \sqrt{x} = x^{1/2}$

② $\begin{cases} u' = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}} \\ du = \dfrac{1}{2\sqrt{x}}\,dx \end{cases}$

③ $\displaystyle 2\int \sin \overbrace{\sqrt{x}}^{u} \cdot \underbrace{\frac{1}{2\sqrt{x}}\,dx}_{du}$

$$= 2\int \sin u \, du$$

④ $= 2(-\cos u) + C$

⑤ $= -2\cos\sqrt{x} + C.$

Ex. $\displaystyle\int \cos^4\theta \sin\theta \, d\theta = \int (\cos\theta)^4 \sin\theta \, d\theta$

① $u = \cos\theta$ inside function, $\cos\theta$ inside $(\ )^4$

② $du = -\sin\theta \, d\theta$

(3) $= -\int -(\cos\theta)^4 \underbrace{\sin\theta \, d\theta}_{du}$

$$= -\int u^4 \, du$$

(4) $= -\dfrac{u^5}{5} + C$

(5) $= -\dfrac{\cos^5\theta}{5} + C$

Ex. $\int u\sqrt{1-u^2} \, du$

Work on this problem on your own

let $w = 1 - u^2$ then $dw = \underline{-2u\,du}$

$$-\frac{1}{2}\int -2\underline{u}\sqrt{1-u^2}\,\underline{du} = -\frac{1}{2}\int \sqrt{w}\,dw$$

$$= -\frac{1}{2}\int w^{1/2}\,dw = -\frac{1}{2}\frac{w^{3/2}}{\frac{3}{2}} + C$$

$$= -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\,w^{3/2} + C$$

$$= -\frac{1}{3}(1-u^2)^{3/2} + C.$$

Definite Integrals with u-sub:

Ex. $\displaystyle\int_0^{\sqrt{\pi}} 3x\sin(x^2)\,dx$ let $u = x^2$

then $du = 2x\,dx$

$$= \frac{3}{2}\int_0^{\sqrt{\pi}} 2x\sin(\overbrace{x^2}^{u})\,dx$$

du

$$= \frac{3}{2}\int_0^{\pi} \sin u \, du$$

when $x = 0$, $u = 0^2 = 0$

when $x = \sqrt{\pi}$ $u = (\sqrt{\pi})^2 = \pi$

* the bounds must match the variable.

$$= \frac{3}{2}(-\cos u)\Big|_0^{\pi}$$

* don't go back to x! move forward with u.

$$= \frac{3}{2}\left[(-\cos\pi) - (-\cos 0)\right]$$

$$= \frac{3}{2}\left[(-(-1)) - (-1)\right]$$

$$= \frac{3}{2}\left[1+1\right] = 3.$$

Ex. $\displaystyle\int_1^3 \cos\left(\frac{\pi t}{2}\right) dt$

STOP Work on this problem on your own

$$u = \frac{\pi t}{2} \quad \text{then} \quad du = \frac{\pi}{2}\,dt$$

$$\frac{2}{\pi}\int_{1}^{3} \frac{\pi}{2}\cos\left(\overbrace{\frac{\pi t}{2}}^{u}\right)dt \;=\; \frac{2}{\pi}\int_{\pi/2}^{3\pi/2} \cos u \, du$$

when $t = 1$, $u = \frac{\pi(1)}{2} = \frac{\pi}{2}$

$t = 3$, $u = \frac{\pi(3)}{2} = \frac{3\pi}{2}$

$$= \frac{2}{\pi}\sin u \Big|_{\pi/2}^{3\pi/2} = \frac{2}{\pi}\left[\left(\sin\frac{3\pi}{2}\right) - \left(\sin\frac{\pi}{2}\right)\right]$$

$$= \frac{2}{\pi}\left[-1 - 1\right] = -\frac{4}{\pi}.$$

Sometimes the substitution requires solving for x in terms of u:

Ex. $-\int -\frac{x^2}{\sqrt{1-x}}\,dx$ $\qquad u = 1-x \Rightarrow x = 1-u$

$\qquad\qquad\qquad\qquad du = -dx$

$$= -\int \frac{(1-u)^2}{u^{1/2}}\,du = -\int \frac{1-2u+u^2}{u^{1/2}}\,du$$

$$= -\int \left(\frac{1}{u^{1/2}} - \frac{2u}{u^{1/2}} + \frac{u^2}{u^{1/2}} \right) du$$

$$= -\int \left(u^{-1/2} - 2u^{1/2} + u^{3/2} \right) du$$

$$= -\left(2u^{1/2} - 2\cdot\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} \right) + C$$

$$= -2u^{1/2} + \frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2} + C$$

$$= -2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C.$$

Sometimes we have a double substitution:

Ex. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}\,\cos^5\sqrt{x}}\,dx$ let u = the innermost function

$= \displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}\,(\cos\sqrt{x})^5}\,dx$

$u = \sqrt{x} = x^{1/2}$

then $du = \frac{1}{2}x^{-1/2}\,dx$

$= \frac{1}{2\sqrt{x}}\,dx$

$= 2\displaystyle\int \frac{\sin\overset{u}{\sqrt{x}}}{\underbrace{2\sqrt{x}}\,(\cos\underset{u}{\sqrt{x}})^5}\,\underline{dx}$ du

$= 2\displaystyle\int \frac{\sin u}{(\cos u)^5}\,du$

still a composition...

let $w = \cos u$

then $dw = -\sin u\,du$

$= -2\displaystyle\int \frac{-\sin u}{(\cos u)^5}\,du$

$$= -2 \int \frac{dw}{w^5} = -2 \int w^{-5}\, dw = -2\, \frac{w^{-4}}{-4} + C$$

$$= \frac{1}{2w^4} + C = \frac{1}{2(\cos u)^4} + C$$

$$= \frac{1}{2(\cos\sqrt{x})^4} + C$$

$$\text{or} = \frac{1}{2\cos^4\sqrt{x}} + C.$$