Math 20100
Calculus I

Lesson 28
The Second Fundamental Theorem of Calculus: The Derivative of an Integral

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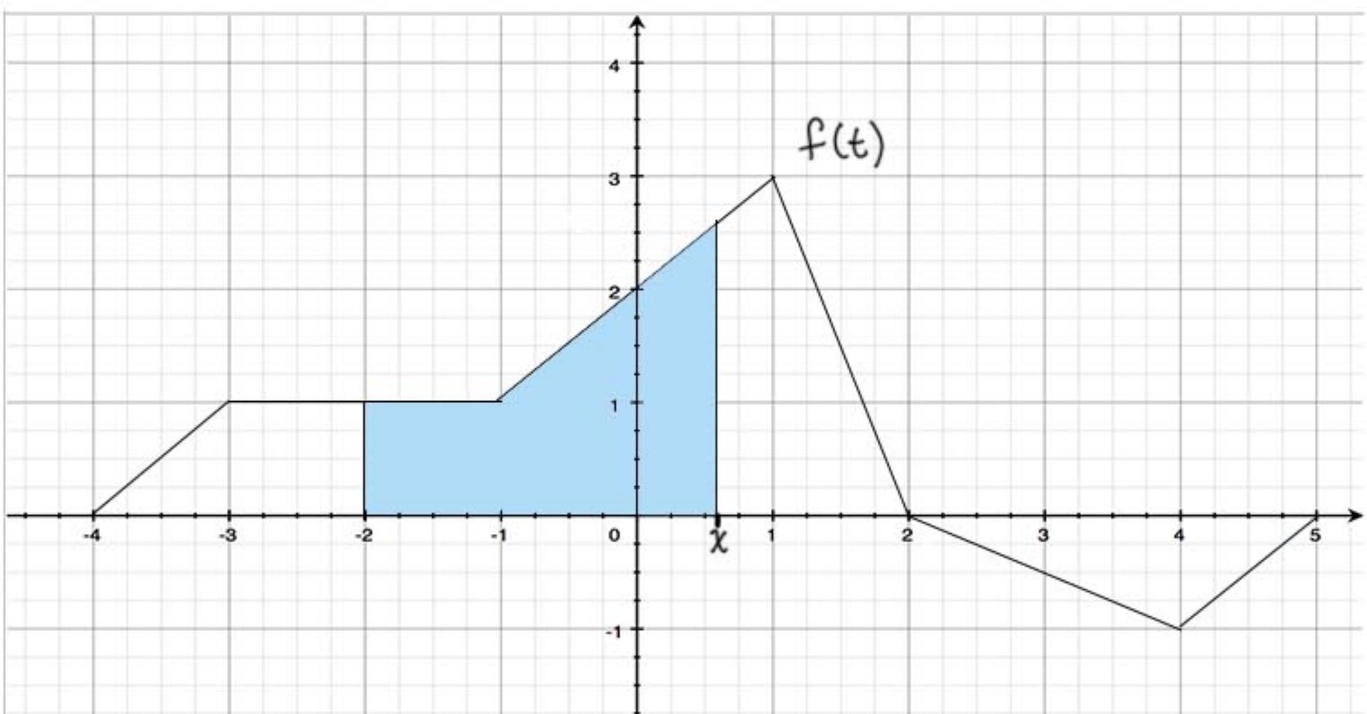
The Second Fundamental Theorem of Calculus: The Derivative of an Integral

Consider the function

$$g(x) = \int_a^x f(t)\,dt$$

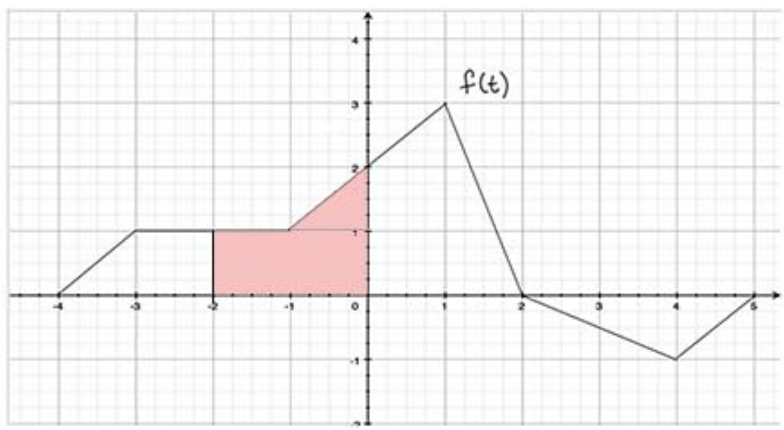
Ex. $$g(x) = \int_{-2}^{x} f(t)\,dt$$



then find $g(-2)$, $g(0)$, $g(2)$, $g(4)$.

$$g(-2) = \int_{-2}^{-2} f(t)\,dt = 0 \quad (\text{no area})$$

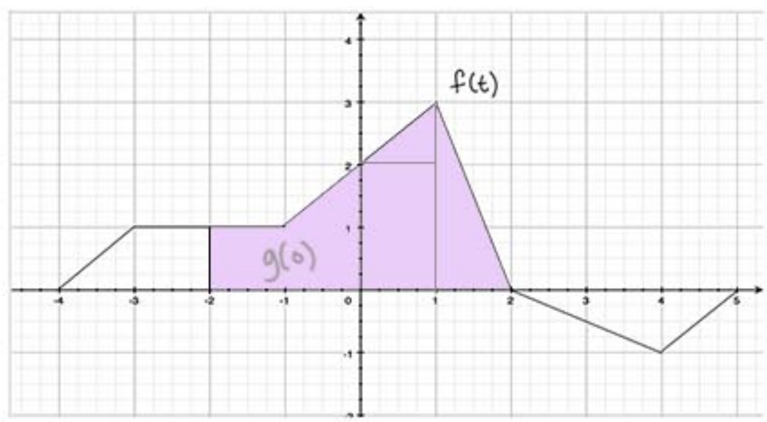
$$g(0) = \int_{-2}^{0} f(t)\,dt$$



$$= (2)(1) + \tfrac{1}{2}(1)(1) = 2\tfrac{1}{2}$$

$$= \tfrac{5}{2}.$$

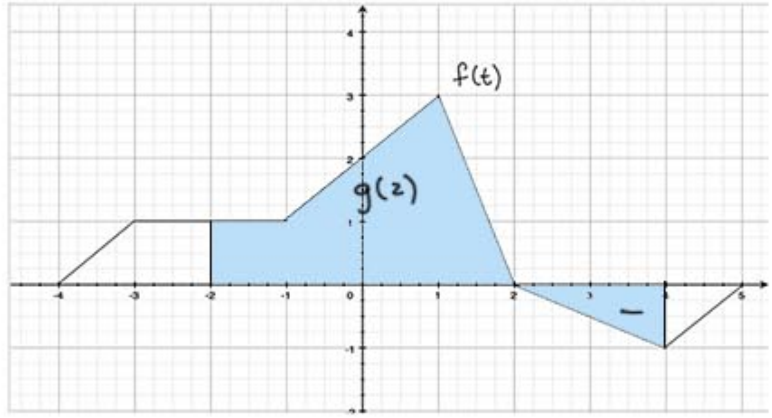
$$g(2) = \int_{-2}^{2} f(t)\,dt$$



$$= g(0) + 1(2) + \tfrac{1}{2}(1)(1) + \tfrac{1}{2}(1)(3) =$$

$$= \tfrac{5}{2} + 2 + \tfrac{1}{2} + \tfrac{3}{2} = \tfrac{13}{2}.$$

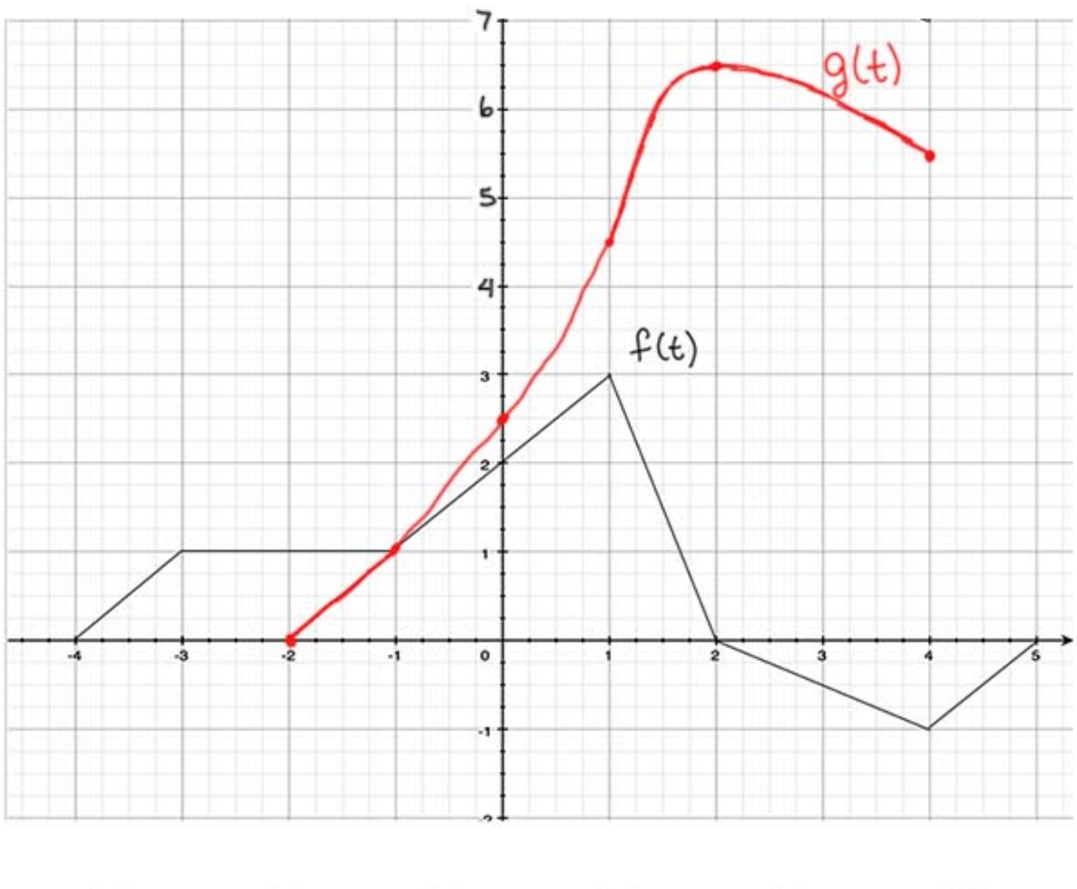
$$g(4) = \int_{-2}^{4} f(t)\,dt$$



$$= g(2) - \tfrac{1}{2}(2)(1)$$

$$= \frac{13}{2} - 1 = \frac{11}{2}.$$

So then a sketch of g on the same set of axes would look like:



Theorem

The Second Fundamental Theorem of Calculus:

If f is continuous on $[a,b]$ and

$$g(x) = \int_a^x f(t)\,dt \quad \text{for } x \in [a,b],$$

then $\quad g'(x) = f(x) \quad$ for $x \in (a,b)$

ie $\quad \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x) \quad$ for $x \in (a,b)$

Ex. $g(x) = \int_0^x \underbrace{(1+\sqrt{t})}_{f(t)}\,dt$

a) find $g'(x)$ by using the second fundamental theorem.

$$g'(x) = f(x) = 1 + \sqrt{x}.$$

b) find $g'(x)$ by first finding an

antiderivative $F(t)$ (by using the first fundamental Theorem)

$$g(x) = \int_0^x (1+\sqrt{t})\,dt = \left[t + \tfrac{2}{3}t^{3/2}\right]_0^x = x + \tfrac{2}{3}x^{3/2}.$$

Then $g'(x) = 1 + \frac{2}{3}\cdot\frac{3}{2}x^{1/2} = 1 + \sqrt{x}$.

Ex. Use the Fundamental Theorem to find $g'(x)$.

$$g(x) = \int_1^x (2+t^4)^5\,dt$$

Work on this problem
on your own

$$g(x) = \int_1^{x} (2+t^4)^5\,dt$$

$$g'(x) = (2+x^4)^5.$$

Ex. Use the Fundamental Theorem to find $g'(x)$.

function of x

$$g(x) = \int_1^{\sin x} (2+t^4)^5 \, dt$$

We need the chain rule.

Can think of this as $u = \sin x$ plugged into

$$h(u) = \int_1^{u} (2+t^4)^5 \, dt$$

$$g'(x) = h'(u) \cdot u'(x) = \left(2+(\sin x)^4\right)^5 \cdot \cos x$$

$$= (\cos x)\left(2+\sin^4 x\right)^5.$$

Ex. Use the Fundamental Theorem to find $g'(x)$.

$$g(x) = \int_2^{1/x} \cos^2 t \, dt$$

STOP Work on this problem on your own

$$g'(x) = \cos^2\left(\frac{1}{x}\right) \cdot \underbrace{\frac{d}{dx}\left(\frac{1}{x}\right)}_{-x^{-2}} = -\frac{1}{x^2}\cos^2\left(\frac{1}{x}\right).$$

Note that from the definition of the definite integral, $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \Delta x = \frac{b-a}{n}$

if we have $\displaystyle\int_b^a f(x)\,dx$ with $b > a$,

then $\Delta x = \dfrac{a-b}{n}$ is negative,

so $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

also $$F(a) - F(b) = -\big(F(b) - F(a)\big).$$

Ex. Use the Fundamental Theorem to find $g'(x)$.

$$g(x) = \int_{\sin x}^{1} \sqrt{1+t^2}\, dt$$

$$= -\int_{1}^{\sin x} \sqrt{1+t^2}\, dt$$

then $g'(x) = -\sqrt{1+\sin^2 x} \cdot \frac{d}{dx}(\sin x)$

$$= -\cos x \sqrt{1+\sin^2 x}\,.$$

The Average Value of a Function

We know how to take the average value of a finite set of numbers, ie for the set $y_1, y_2, \dots y_n$ we have $\text{avg} = \dfrac{y_1+y_2+\cdots+y_n}{n}$.

But what does it mean to take an average value of f over the interval $[a,b]$?

Start with a finite number of function evaluations from equally spaced subintervals:

$$\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n}$$

where $c_i \in [x_{i-1}, x_i]$ the i^{th} subinterval

and $\Delta x = \frac{b-a}{n}$ is the width of the subintervals

$\therefore n = \frac{b-a}{\Delta x}$ and we have

$$\frac{(f(c_1) + f(c_2) + \cdots + f(c_n))\Delta x}{b-a} = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x$$

But this is only an average of finitely many function evaluations. To get an average over all of $[a,b]$, take the limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$\therefore \text{average value of } f \text{ over } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Also, using the Mean Value Theorem we can show that for f continuous on $[a,b]$, this average value is $f(c)$ for some $c \in (a,b)$:

If f is continuous on $[a,b]$, then

$$g(x) = \int_a^x f(t)\,dt \text{ is continuous on } [a,b].$$

We know by the Fundamental Theorem that

$g(x)$ is differentiable on (a,b).

$$\therefore \exists\, c \in (a,b) \ni g'(c) = \frac{g(b) - g(a)}{b-a}$$

$$f(c) = \frac{\int_a^b f(t)\,dt - 0}{b-a}$$

$$\therefore f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Ex. Find the average value of $f(x) = x^2$ on $[0,2]$.

Also find the value of $c \in (0,2)$ for which $f(c) =$ the average value.

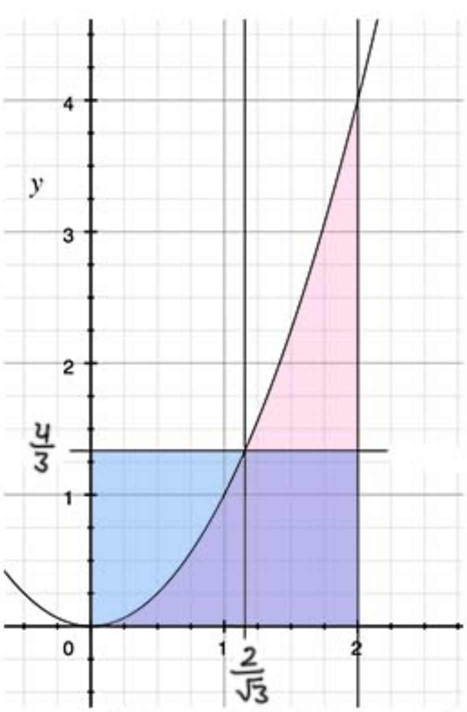
average value of f over $[0,2]$ $= \dfrac{1}{2-0} \displaystyle\int_0^2 x^2\,dx = \dfrac{1}{2} \dfrac{x^3}{3} \Big|_0^2 =$

$$= \frac{1}{2}\left(\frac{8}{3}\right) = \frac{4}{3}.$$

To find c: $f(c) = \frac{4}{3} \Rightarrow c^2 = \frac{4}{3}$

$c = \pm\sqrt{\frac{4}{3}}$ only $+\sqrt{\frac{4}{3}} \in (0,2)$

$c = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}.$



notice: the area of the rectangle is equal to the area under the curve

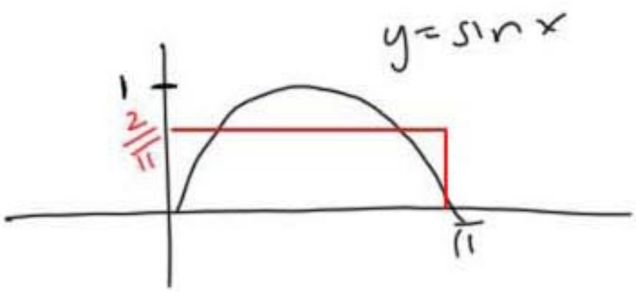
$$\left(\frac{4}{3}\right)(2) = \frac{8}{3} = \int_0^2 x^2\,dx.$$

Ex. find the average value of $y = \sin x$ on $[0, \pi]$.

$$\text{avg value of } f \text{ over } [0,\pi] = \frac{1}{\pi - 0}\int_0^{\pi} \sin x \, dx = \frac{1}{\pi}\left(-\cos x\right)\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\cos\pi\right) - \left(-\cos 0\right)\right]$$

$$= \frac{1}{\pi}\left[(+1) - (-1)\right] = \frac{2}{\pi} \approx \frac{2}{3}$$

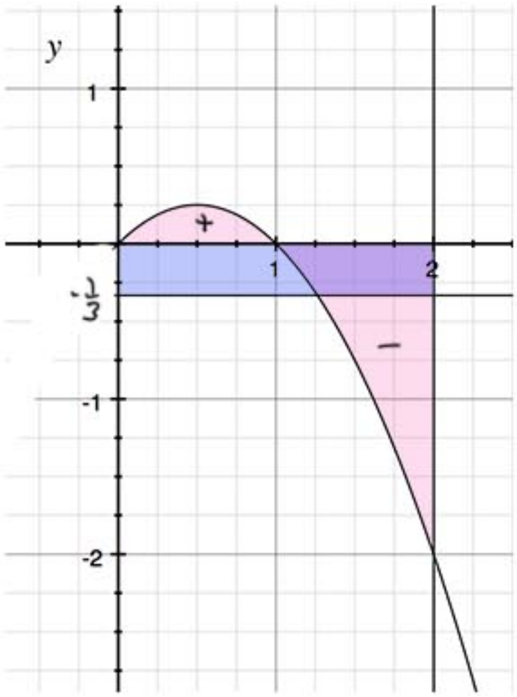


Ex. find the average value of $f(x) = x - x^2$ over $[0, 2]$

$$\frac{1}{2-0}\int_0^2 \left(x - x^2\right) dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^2$$

$$= \frac{1}{2}\left[\left(\frac{2^2}{2} - \frac{2^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right)\right]$$

$$= \frac{1}{2}\left(2 - \frac{8}{3}\right) = \frac{1}{2}\left(\frac{6}{3} - \frac{8}{3}\right) = \frac{1}{2}\left(-\frac{2}{3}\right) = \boxed{-\frac{1}{3}}$$



area of rectangle =

<u>net</u> area under curve

given by the integral