Math 20100
Calculus I

Lesson 26
The First Fundamental Theorem of Calculus: Evaluating Definite Integrals

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The First Fundamental Theorem of Calculus:

Evaluating Definite Integrals

In the last two lessons we've seen a preview to this short cut for evaluating $\int_a^b f(x)\,dx$.

<u>Theorem</u>

The First fundamental Theorem of Calculus (also called The Evaluation Theorem):

If f is continuous on $[a,b]$,

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where}$$

F is any antiderivative of f.

Proof: we'll start with $F(b)-F(a)$ and show it equals $\int_a^b f(x)\,dx$.

$$F(b)-F(a) = F(x_n) - F(x_0) =$$

partition $[a,b] = x_0, x_1, \dots x_n$ with $x_0 = a$, $x_n = b$

with $\Delta x = \frac{b-a}{n}$

$$F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + F(x_{n-2}) + \cdots - F(x_2) + F(x_2) - F(x_1) + F(x_1) - F(x_0)$$

$$= \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

now use Mean Value Theorem:

if F continuous on $[a,b]$ and differentiable on (a,b), then

$$\exists\, c \in (a,b) \ni F'(c) = \frac{F(b)-F(a)}{b-a}.$$

Use on interval $[x_{i-1}, x_i]$ with $F(x)$.

then $\exists\, c_i \in (x_{i-1}, x_i) \ni$

$$\underbrace{F'(c_i)}_{f(c_i)} = \frac{F(x_i)-F(x_{i-1})}{\underbrace{x_i - x_{i-1}}_{\Delta x}}$$

$$\therefore\ \exists\, c_i \in (x_{i-1}, x_i) \ni f(c_i)\Delta x = F(x_i) - F(x_{i-1})$$

so far $F(b)-F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$

$$= \sum_{i=1}^{n} f(c_i)\Delta x \quad \text{true } \forall n$$

so $\lim\limits_{n \to \infty} \left(F(b) - F(a) \right) = \lim\limits_{n \to \infty} \underbrace{\sum_{i=1}^{n} f(c_i) \Delta x}_{\text{Riemann Sum}}$

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

Ex. $\displaystyle\int_1^8 \sqrt[3]{x}\,dx =$

$$= \int_1^8 x^{1/3}\,dx = \left(\frac{x^{4/3}}{\frac{4}{3}} \right) \Bigg|_1^8 =$$

$$= \left(\underbrace{\frac{3}{4} x^{4/3}}_{F(x) \big|_1^8} \right) \Bigg|_1^8 = \underbrace{\left(\frac{3}{4} \cdot 8^{4/3} \right)}_{\substack{x=8 \\ F(b)}} - \underbrace{\left(\frac{3}{4} 1^{4/3} \right)}_{\substack{x=1 \\ F(a)}}$$

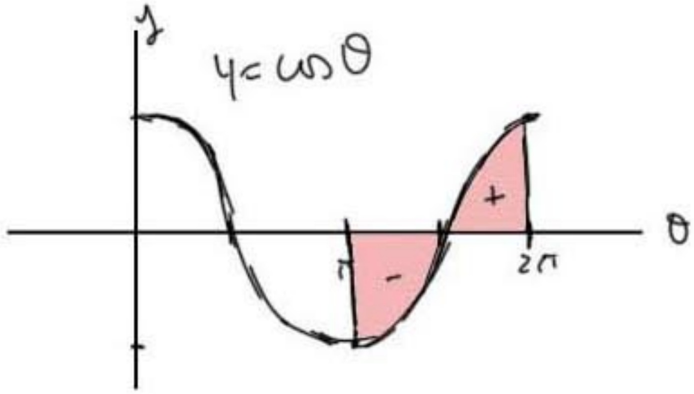
$$= \left(\frac{3}{4} 2^4 \right) - \frac{3}{4}$$

$$= 12 - \frac{3}{4} = \frac{48}{4} - \frac{3}{4} = \boxed{\frac{45}{4}}$$

Ex. $\displaystyle\int_{\pi}^{2\pi} \cos\theta \, d\theta =$

$$= (\sin\theta)\Big|_{\pi}^{2\pi}$$

$$= (\sin 2\pi) - (\sin \pi) = 0 - 0 = 0.$$



Ex. $\displaystyle\int_{1}^{9} \frac{3x-2}{\sqrt{x}}\, dx$

$$= \int_{1}^{9} \left(\frac{3x}{\sqrt{x}} - \frac{2}{\sqrt{x}} \right) dx = \int_{1}^{9} \left(3x^{1/2} - 2x^{-1/2} \right) dx$$

$$= \left(\frac{3x^{3/2}}{\frac{3}{2}} - 2\frac{x^{1/2}}{\frac{1}{2}} \right)\Big|_{1}^{9} = \left(3 \cdot \frac{2}{3} x^{3/2} - 2 \cdot \frac{2}{1} x^{1/2} \right)\Big|_{1}^{9}$$

$$= \left(2x^{3/2} - 4x^{1/2}\right)\Big|_1^9 =$$

$$= \left(2\cdot 9^{3/2} - 4\cdot 9^{1/2}\right) - \left(2\cdot 1^{3/2} - 4\cdot 1^{1/2}\right)$$

$$\quad 2\cdot 27 - 12 \qquad \underbrace{-(2-4)}_{+2}$$

$$= 54 - 10 = \boxed{44}.$$

Ex. $$\int_{\pi/4}^{\pi/3} \frac{\sec\theta + \tan\theta}{\cos\theta}\, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \left(\frac{\sec\theta}{\cos\theta} + \frac{\tan\theta}{\cos\theta}\right) d\theta = \int_{\pi/4}^{\pi/3} \left(\sec^2\theta + \sec\theta\tan\theta\right) d\theta$$

$$= \left[\tan\theta + \sec\theta\right]\Big|_{\pi/4}^{\pi/3} =$$

$$= \left(\tan\tfrac{\pi}{3} + \sec\tfrac{\pi}{3}\right) - \left(\tan\tfrac{\pi}{4} + \sec\tfrac{\pi}{4}\right) =$$

$$= \left(\sqrt{3} + 2\right) - \left(1 + \sqrt{2}\right) = 1 - \sqrt{2} + \sqrt{3}.$$

$$\tan\frac{\pi}{3} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \qquad \sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$$

$$\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \qquad \sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Applications of The Definite Integral:

$$\int_a^b \underbrace{\text{rate of change function}}_{f'(t)\,dt} = \text{total change over } [a,b].$$

$$f(b) - f(a)$$

Ex. If $f(x)$ represents the slope of a trail at a (horizontal) distance of x miles from the start of the trail, what does $\int_3^5 f(x)\,dx$ represent?

$f(x)$ = slope of trail = rate of change of elevation

$$= \frac{\text{change in elevation}}{\text{change in distance } x}$$

then $\int_3^5 f(x)\,dx$ = total change in elevation from $x = 3$ mi to $x = 5$ mi.

Ex. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \le t \le 50$. Find the amount of water that flows from the tank during the first 10 minutes.

$r(t)$ = rate of change function

we're asked for total change in the first 10 min

$$\int_0^{10} r(t)\,dt = \int_0^{10} (200 - 4t)\,dt = \left[200t - \frac{4t^2}{2}\right]_0^{10}$$

$$= \left[200t - 2t^2\right]_0^{10} = \left(200(10) - 2(10)^2\right) - (0 - 0) =$$

$$= 2000 - 200 = 1800 \text{ liters.}$$

Notice here that the rate of change here

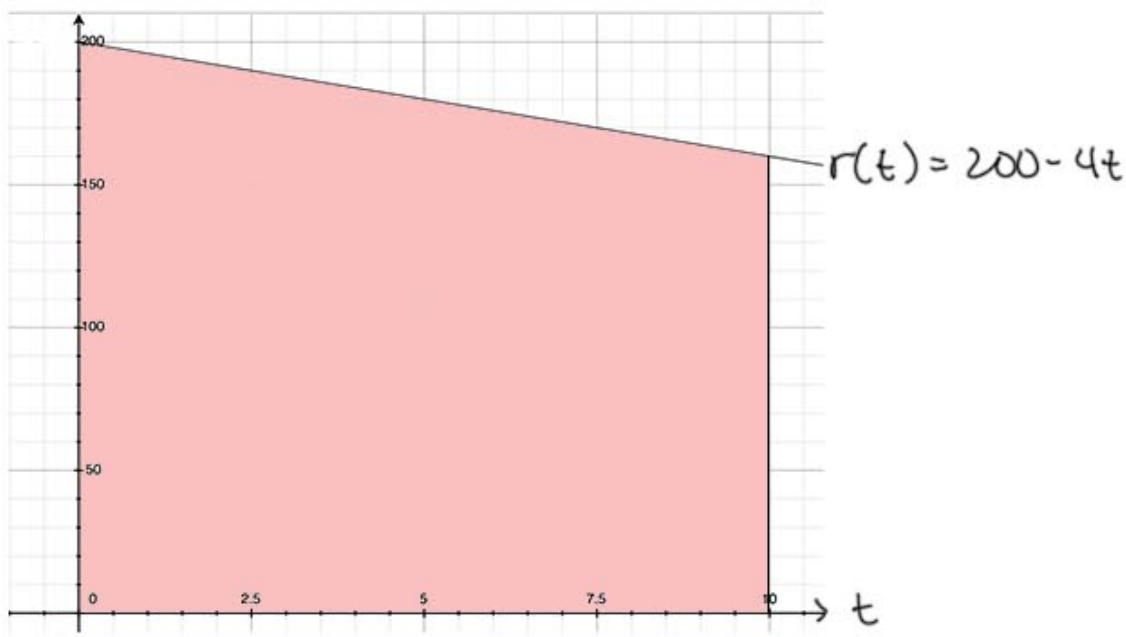
was always positive, $r(t) = 200 - 4t \quad 0 \le t \le 10$.

This says the water was always flowing the

same direction, and that the total amount

of water flowing = the definite integral.

ie, the area is above the t axis:



Sometimes we need to be careful to distinguish

between the total (net) change given by the

definite integral, and the total amount of area:

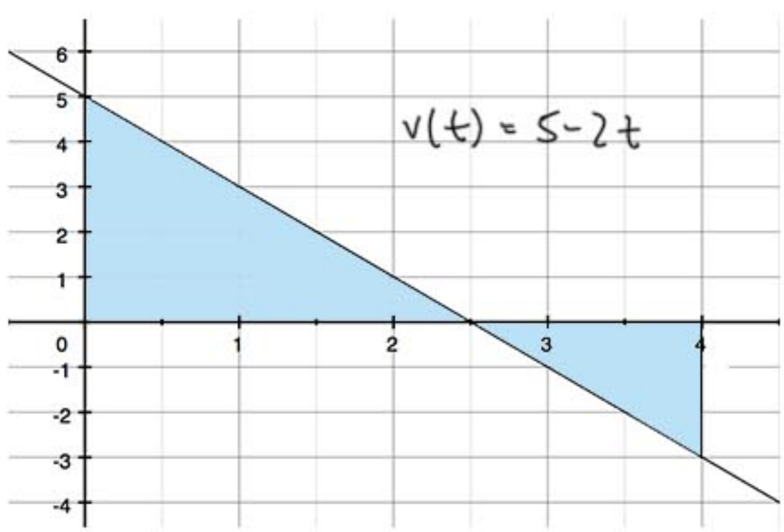
Ex. $v(t) = 5 - 2t$ is the velocity function (in ft/s) for a particle moving along a line.

a) Find the displacement of the particle during the time interval $0 \le t \le 4$

(displacement = final position − initial position)

b) Find the distance traveled by the particle during the time interval $0 \le t \le 4$.

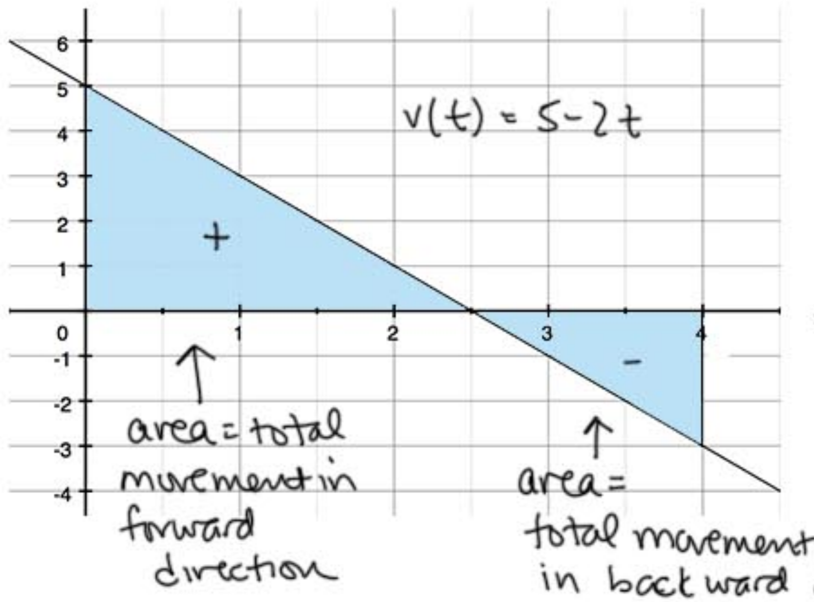
(adds up the distance traveled in both directions)



a) $\int_0^4 v(t)\,dt = \int_0^4 (5-2t)\,dt = \left[5t - t^2\right]_0^4 =$

$= \left(5(4)-4^2\right) - (0-0) = 20 - 16 = 4\text{ ft.}$

displacement.



net area:

$\frac{1}{2}(2.5)(5) - \frac{1}{2}(1.5)(3)$

$= \frac{1}{2}\left(\frac{5}{2}\right)5 - \frac{1}{2}\left(\frac{3}{2}\right)(3)$

$= \frac{25}{4} - \frac{9}{4} = \frac{16}{4} = 4.$

b) we need to include both areas with a positive sign.

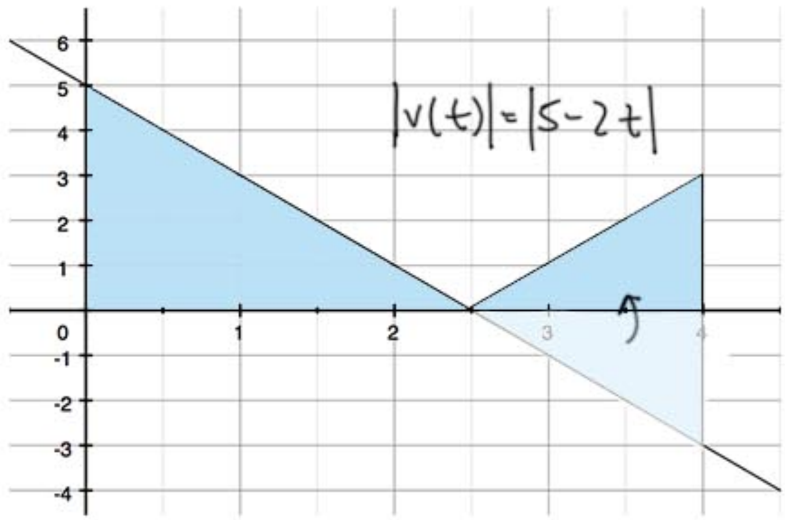
note: 1) we need to break up the integral at $t = 2.5$

and take $\int_0^{2.5} v(t)\,dt - \underbrace{\int_{2.5}^4 v(t)\,dt}_{\text{negative}}$

$-(-)$ makes it positive

$$= \int_0^{2.5} (5-2t)\,dt - \int_{2.5}^{4} (5-2t)\,dt$$

2) this is $\int_0^4 |v(t)|\,dt$



$$= \int_0^4 |5-2t|\,dt = \int_0^{2.5} (5-2t)\,dt + \int_{2.5}^{4} -(5-2t)\,dt$$

since $|5-2t| = \begin{cases} 5-2t & \text{when } 5-2t \geq 0 \text{ ie } t \leq \frac{5}{2} \\ -(5-2t) & \text{when } 5-2t < 0 \text{ ie } t > \frac{5}{2} \end{cases}$

either way, 1) or 2), we have :

$$\left[5t - t^2\right]_0^{2.5} - \left[5t - t^2\right]_{2.5}^{4} = \qquad 2.5 = \frac{5}{2}$$

$$= \left(5\left(\frac{5}{2}\right) - \left(\frac{5}{2}\right)^2\right) - (0 - 0) - \left[\left(5(4) - 4^2\right) - \left(5\left(\frac{5}{2}\right) - \left(\frac{5}{2}\right)^2\right)\right]$$

$$= \frac{25}{2} - \frac{25}{4} - \left[4 - \left(\frac{25}{2} - \frac{25}{4}\right)\right] =$$

$$= \frac{25}{2} - \frac{25}{4} - 4 + \frac{25}{2} - \frac{25}{4} = \frac{50}{2} - \frac{50}{4} - 4$$

$$= \frac{50}{2} - \frac{25}{2} - 4 = \frac{25}{2} - \frac{8}{2} = \frac{17}{2} \text{ ft.}$$

distance traveled.