Math 20100
Calculus I

Lesson 25
The Definite Integral

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The Definite Integral

In lesson 24 we saw:

left hand sum:

$$\text{area} \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

right hand sum:

$$\text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

And in these area approximations we are using $f(x_{i-1})$ or $f(x_i)$ as the height of the rectangle over the i^{th} subinterval.

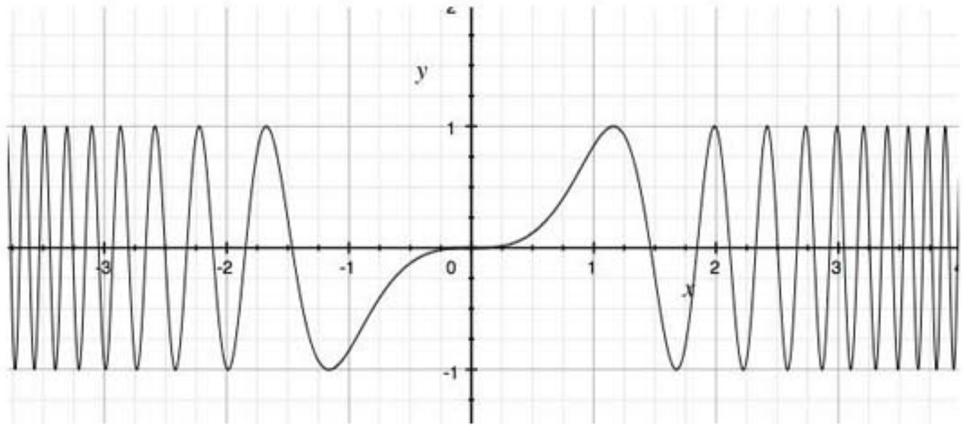
Actually, we can use any $c_i \in [x_{i-1}, x_i]$ ← i^{th} subinterval

any point in the subinterval

and take $f(c_i)$ as the height of the rectangle over the i^{th} subinterval.

Also, we could use different sizes of Δx for the different subintervals, ie Δx_i = size of i^{th} subinterval.

Ex. $f(x) = \sin(x^3)$



for this function it would be useful to have an adaptable subinterval width. We would use larger subintervals say from $[-1,1]$ where there is less function variation, and smaller subintervals outside of $[-1,1]$.

<u>Definite Integral</u> of f from $x=a$ to $x=b$:

$$\int_a^b f(x)\,dx = \underbrace{\lim_{\max \Delta x_i \to 0}} \overbrace{\sum_{i=1}^{n} f(c_i)\,\Delta x_i}^{\text{Riemann Sum}}$$

width of the i^{th} subinterval

$c_i \in \underbrace{[x_{i-1}, x_i]}_{i^{th} \text{ subinterval}}$

replaces $\lim\limits_{n\to\infty}$ from last time because now can have varying Δx_i sizes.

$f(c_i)$ height of i^{th} rectangle

Notice, then, if f is integrable (the limit exists),

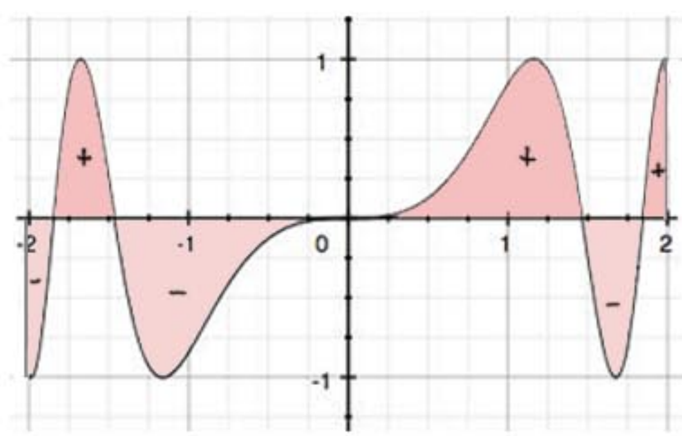
then $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

this gave us exact area under $f(x)$ over $[a,b]$

Note: we've been saying "under the curve..." and that really only applies to functions $f(x) \geq 0$ on $[a,b]$.

What does the integral computation give for functions with values above + below the x-axis?

Ex. $$\int_{-2}^{2} \sin(x^3)\,dx$$



In the Riemann sums, when $f(c_i) < 0$ the area has a negative sign. So in the integral, areas above the x-axis are counted with a positive sign, and areas below the x-axis are counted with a negative sign. So the integral in this case does not measure total area (as in the amount of carpet we'd need to cover the shaded area).

We'll see an example toward the end of this lesson.

Ex. $\int_0^5 (1+2x^3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

$f(x) = 1+2x^3$

$a = 0$
$b = 5$ $\quad \Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \boxed{\frac{5}{n}}$

$x_i = a + i\Delta x$

$= 0 + i\left(\frac{5}{n}\right) \Rightarrow f(x_i) = 1+2\left(\frac{5i}{n}\right)^3$

$= \frac{5i}{n}$ $\qquad = 1 + 2\cdot\frac{125 i^3}{n^3}$

$= 1 + \frac{250 i^3}{n^3}$

$$\int_0^5 (1+2x^3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1+250\frac{i^3}{n^3}\right)\left(\frac{5}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5}{n} + 1250\frac{i^3}{n^4}\right)$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{5}{n} + \frac{1250}{n^4} \sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty} \left[\frac{5}{\cancel{n}}\cdot\cancel{n} + \frac{1250}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$= 5 + \frac{1250}{4}$$

$$= \frac{10}{2} + \frac{625}{2} = \boxed{\frac{635}{2}}$$

preview from lesson 26:

$$\int_0^5 (1+2x^3)\,dx = \left(x + \cancel{2}\frac{x^4}{\cancel{4}}\right)\Bigg|_0^5 =$$

$$= \left(5 + \frac{5^4}{2}\right) - \left(0 + \frac{0^4}{2}\right) = 5 + \frac{625}{2} = \boxed{\frac{635}{2}}$$

$$\text{Ex. } \int_1^2 x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\underbrace{\Delta x}_{\frac{1}{n}}$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x$$

$$= 1 + i\left(\frac{1}{n}\right)$$

$$f(x) = x^3$$

$$x_i = 1 + \frac{i}{n} \quad \Rightarrow f(x_i) = \left(1+\frac{i}{n}\right)^3$$

$$\int_1^2 x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{i}{n}\right)^3 \frac{1}{n}$$

$$\left(1+\frac{i}{n}\right)^3 =$$

$$\left(1+\frac{i}{n}\right)\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)$$

$$= 1 + \frac{2i}{n} + \frac{i^2}{n^2}$$

$$+ \frac{i}{n} + 2\frac{i^2}{n^2} + \frac{i^3}{n^3}$$

$$= 1 + \frac{3i}{n} + 3\frac{i^2}{n^2} + \frac{i^3}{n^3}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{3i}{n}+\frac{3i^2}{n^2}+\frac{i^3}{n^3}\right)\frac{1}{n}$$

$$\frac{1}{n} + \frac{3i}{n^2} + \frac{3i^2}{n^3} + \frac{i^3}{n^4}$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{n} + \sum_{i=1}^{n}\frac{3i}{n^2} + \sum_{i=1}^{n}\frac{3i^2}{n^3} + \sum_{i=1}^{n}\frac{i^3}{n^4}\right]$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{n} + \frac{3}{n^2} \sum_{i=1}^{n} i + \frac{3}{n^3} \sum_{i=1}^{n} i^2 + \frac{1}{n^4} \sum_{i=1}^{n} i^3 \right]$$

$$= \lim_{n\to\infty} \left[\cancel{n \cdot \frac{1}{n}} + \frac{3}{n^2} \frac{n(n+1)}{2} + \frac{3}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4} \frac{n^2(n+1)^2}{4} \right]$$

$$= 1 + \frac{3}{2} + \cancel{\frac{4}{6}}1 + \frac{1}{4}$$

$$= 2 + \frac{6}{4} + \frac{1}{4} = 2 + \frac{7}{4} = 3\frac{3}{4} \text{ or } \boxed{\frac{15}{4}}$$

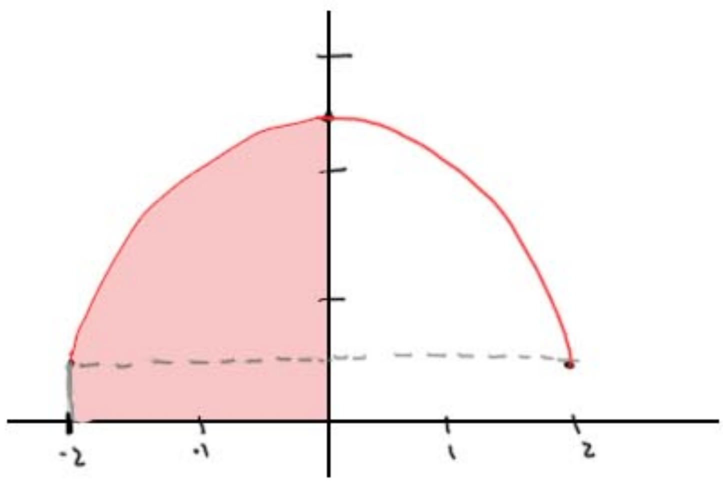
Check: $\displaystyle\int_1^2 x^3\,dx = \left.\frac{x^4}{4}\right]_1^2 = \frac{2^4}{4} - \frac{1^4}{4}$

$$\frac{16}{4} - \frac{1}{4} = \boxed{\frac{15}{4}}$$

We said above that $\displaystyle\int_a^b f(x)\,dx$ can represent areas above + below the x-axis, and so sometimes it's easier to compute a definite integral by interpreting it as an area:

Ex. $\displaystyle\int_{-2}^{0} \underbrace{\left(\tfrac{1}{2} + \sqrt{4-x^2}\right)}_{f(x)} dx = \tfrac{1}{4}$ area of circle + area of rectangle

semicircle of radius 2 moved up 1/2 unit

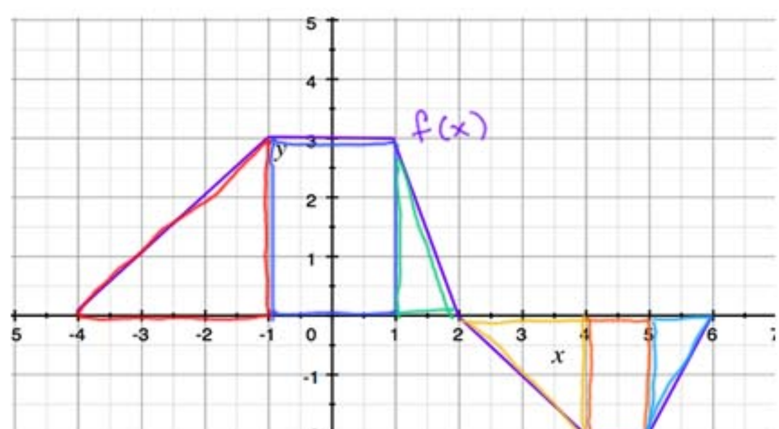


$= \tfrac{1}{4}\pi(2)^2 + \tfrac{1}{2}(2)$

$= \pi + 1$.

Ex. Use areas to compute

$\displaystyle\int_{-4}^{6} f(x)\, dx$.



$$\int_{-4}^{6} f(x)\,dx = \int_{-4}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx + \int_{4}^{5} f(x)\,dx + \int_{5}^{6} f(x)\,dx$$

$$= \tfrac{1}{2}(3)(3) + (2)(3) + \tfrac{1}{2}(1)(3) - \tfrac{1}{2}(2)(2) - 1(2) - \tfrac{1}{2}(1)(2) =$$

$$= \tfrac{9}{2} + 6 + \tfrac{3}{2} - 2 - 2 - 1 = 12 - 5 = 7.$$