Math 20100
Calculus I

Lesson 24
Area Under a Curve

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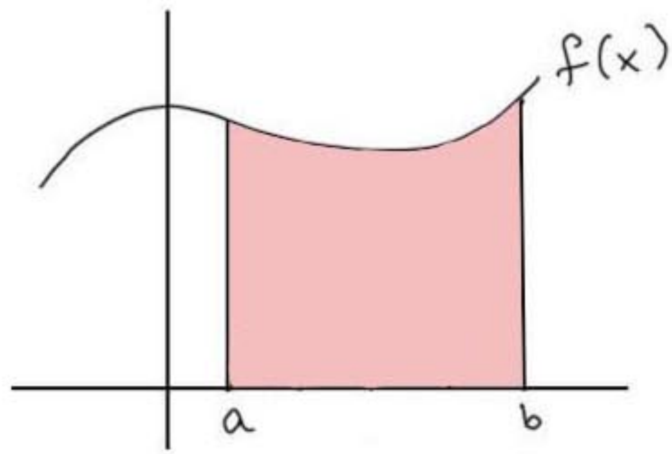
Area Under a Curve

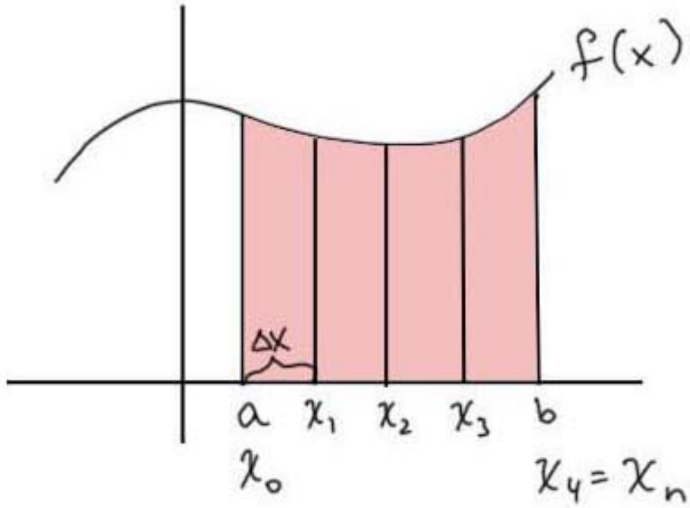
In This Lesson we learn how to find the area under the graph of $y = f(x)$ over the interval $[a, b]$:



$(f(x) \geq 0)$

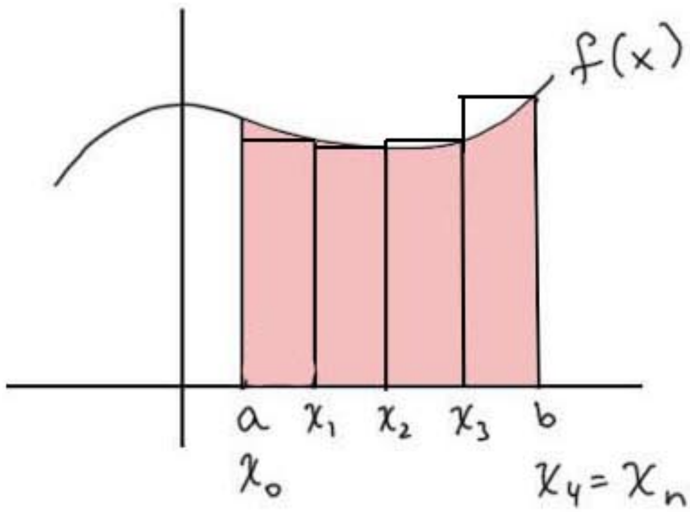
We will approximate The area under curve by dividing $[a, b]$ into n subintervals and using a rectangle to approximate the area on each subinterval. To get The exact area, we'll take The limit as The number of rectangles approaches infinity.

the area divided into subintervals :



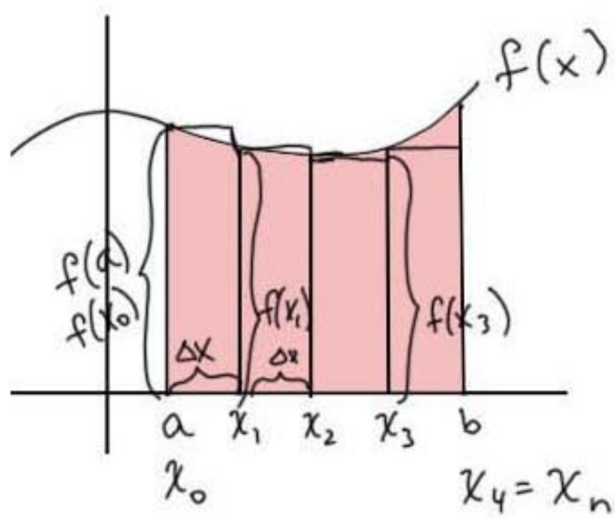
$\Delta x = \frac{b-a}{n}$

we'll find the area of an approximating rectangle on each subinterval



Left hand sum :

for the height of the rectangle, we use the function value on the left of the subinterval



Area under curve $\approx f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$

$$= \sum_{i=0}^{3} f(x_i)\,\Delta x \quad \text{or} \quad = \sum_{\underbrace{i=1}_{i^{\text{th}}\text{ subinterval}}}^{4} f(x_{i-1})\,\Delta x$$

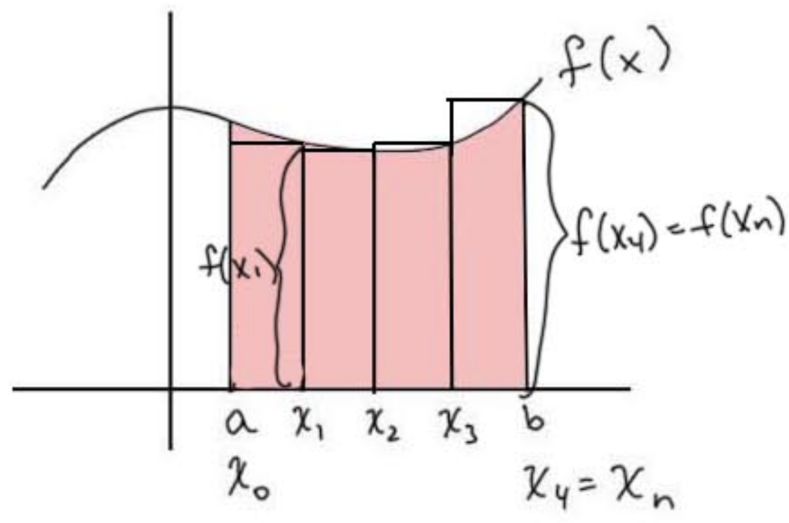
4 ← n = number of subintervals

in general, the left hand sum has

$$\text{Area} \approx \sum_{i=1}^{n} f(x_{i-1})\,\Delta x.$$

Right hand sum:

for the height of the rectangle, we use the function value on the right of the subinterval



area under curve $\approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

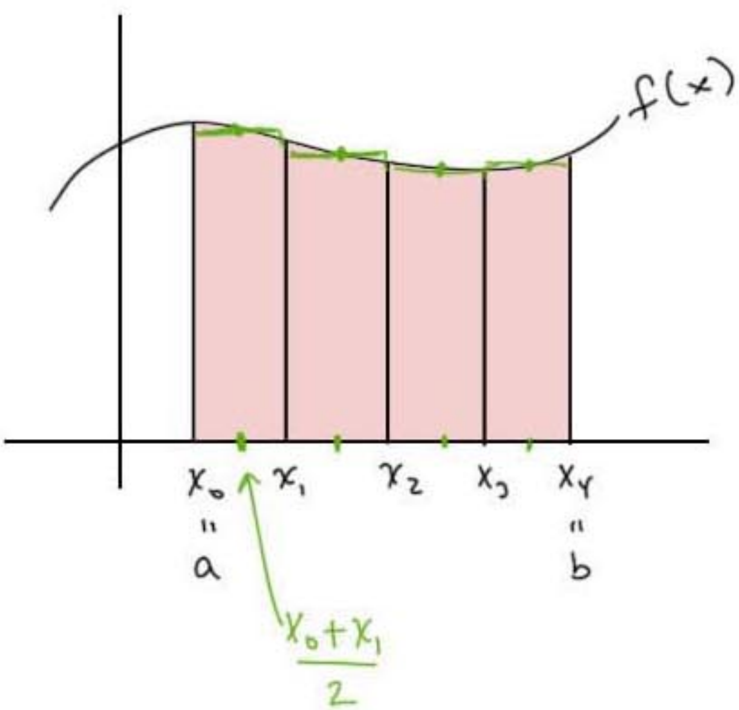
$$= \sum_{i=1}^{4} f(x_i)\,\Delta x$$

In general, the right hand sum has

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

We could also use other "rules" to approximate, for example the Midpoint Rule uses the function evaluation on the midpoint of each subinterval:

Midpoint Rule uses the function value at the midpoint of each subinterval as the height of rectangle.



Midpoint Rule:

$$\text{area} \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

Notice that with any of the rules above, as we increase the number of rectangles, we get a better approximation. So the exact area

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

(notice this uses a right hand sum, that is only for ease of notation. Can use left hand sum or midpoint, same area)

for finding Exact area using $\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$

<u>steps</u> ① $\underbrace{\text{find } \Delta x}_{\text{in terms of } n}$ $\quad \Delta x = \frac{b-a}{n}$

② $\underbrace{\text{find } x_i}_{\text{in terms of } i \text{ and } n}$ $\quad x_i = a + i\Delta x$

③ $\underbrace{\text{find } f(x_i)}_{\text{in terms of } i \text{ and } n}$ by plugging x_i into $f(x)$

④ set up $\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i)}_{\substack{\uparrow \\ \text{plug in} \\ f(x_i) \text{ from step ③}}} \underbrace{\Delta x}_{\text{plug in } \Delta x \text{ from step ①}}$

⑤ simplify/separate into summations on i, i^2, i^3 etc.
leave the $\lim_{n\to\infty}$ outside it all

⑥ use summation formulas $\left(\text{Ex. } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right)$
to get everything in terms of n

⑦ simplify and take limit as $n \to \infty$.

Ex. $f(x) = 4 - x^2$ on $[-1, 1]$.

① sketch the curve and shade the area under $f(x)$ over $[-1, 1]$.

② Approximate the area using a left hand sum with $n = 4$

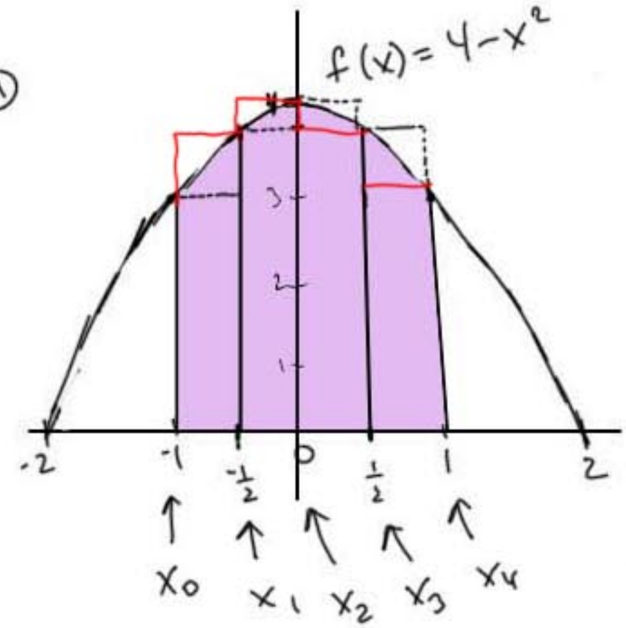
③ Approximate the area using a right hand sum with $n = 4$

④ Approximate the area using the midpoint rule with $n=4$

⑤ Find the exact area by finding the limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

①



for all approximations,

$$\Delta x = \frac{b-a}{n} = \frac{1--1}{4} = \frac{2}{4} = \frac{1}{2}$$

② left hand sum: $\text{area} \approx \frac{1}{2}\left(f(-1)+f(-\tfrac{1}{2})+f(0)+f(\tfrac{1}{2})\right)$

$f(-\frac{1}{2}) = 4-\left(\frac{1}{2}\right)^2$

$4-\frac{1}{4} = 3\frac{3}{4} = \frac{15}{4}$

$$= \frac{1}{2}\left(3+\frac{15}{4}+4+\frac{15}{4}\right)$$

$$= \frac{1}{2}\left(7+\frac{30}{4}\right)$$

$$\frac{1}{2}\left(\frac{14}{2}+\frac{15}{2}\right) = \frac{29}{4}.$$

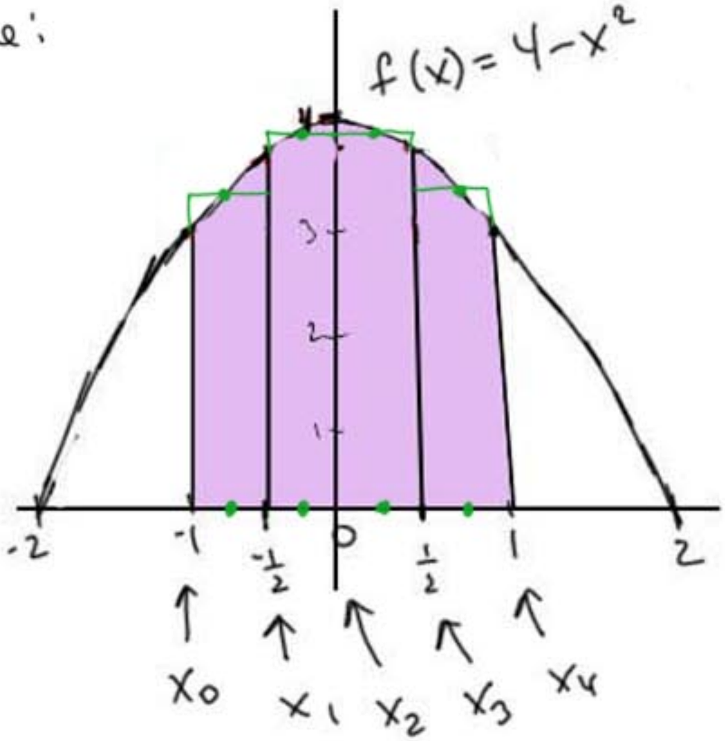
③ right hand sum:

$$\text{area} \approx \frac{1}{2}\left(f(-\tfrac{1}{2}) + f(0) + f(\tfrac{1}{2}) + f(1)\right)$$

$$= \frac{1}{2}\left(\frac{15}{4} + 4 + \frac{15}{4} + 3\right) = \frac{29}{4}.$$

same as left
because of symmetry of the function

④ midpoint rule:



$$\text{area} \approx \frac{1}{2}\left(f(-\tfrac{3}{4}) + f(-\tfrac{1}{4}) + f(\tfrac{1}{4}) + f(\tfrac{3}{4})\right)$$

$$f(\pm\tfrac{1}{4}) = 4 - \frac{1}{16} = \frac{63}{16}$$

$$f(\pm\tfrac{3}{4}) = 4 - \frac{9}{16} = \frac{55}{16}$$

$$= \frac{1}{2}\left(\frac{55}{16} + \frac{63}{16} + \frac{63}{16} + \frac{55}{16}\right)$$

$$= \frac{1}{2}\left(\frac{236}{16}\right) = \frac{118}{16} = \frac{59}{8}.$$

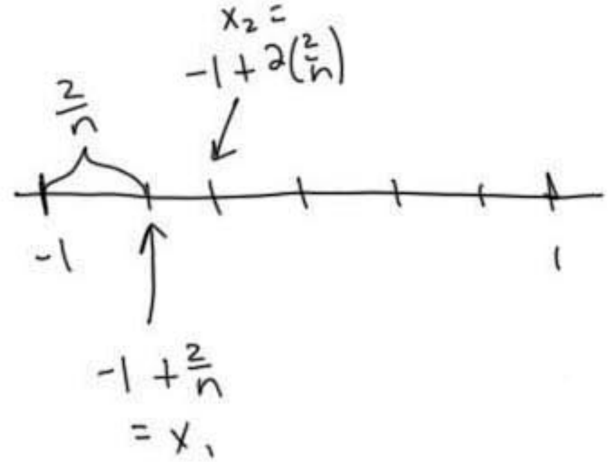
⑤ exact area $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\underbrace{\Delta x}_{\frac{2}{n}}$

$$\Delta x = \frac{b-a}{n} = \frac{1--1}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x$$

$$x_i = -1 + i\left(\frac{2}{n}\right)$$

$$= -1 + \frac{2i}{n}$$



we need $f(x_i) = f\left(-1 + \frac{2i}{n}\right) = 4 - \left(-1 + \frac{2i}{n}\right)^2$

$$= 4 - \left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) = 3 + \frac{4i}{n} - \frac{4i^2}{n^2}$$

$$\text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3 + \frac{4i}{n} - \frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

or pull out of summation since $\frac{2}{n}$ is constant

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{6}{n} + \frac{8i}{n^2} - \frac{8i^2}{n^3} \right)$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{6}{n} + \sum_{i=1}^{n} \boxed{\frac{8i}{n^2}} - \sum_{i=1}^{n} \boxed{\frac{8i^2}{n^3}} \right]$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{6}{n} + \frac{8}{n^2} \sum_{i=1}^{n} i - \frac{8}{n^3} \sum_{i=1}^{n} i^2 \right]$$

$$= \lim_{n\to\infty} \left[\cancel{n} \cdot \frac{6}{\cancel{n}} + \frac{8}{n^2} \frac{n(n+1)}{2} - \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} \right]$$

$$= 6 + 4 - \frac{8}{3} = 10 - \frac{8}{3} = \frac{30}{3} - \frac{8}{3} = \boxed{\frac{22}{3}}$$

exact area.

Preview from lesson 26:

$$\text{area} = \int_{-1}^{1} (4 - x^2)\, dx = \left(4x - \frac{x^3}{3} \right) \Bigg|_{-1}^{1}$$

$$= \left(4(1) - \frac{1}{3} \right) - \left(4(-1) - \left(-\frac{1}{3}\right) \right)$$

$$= \frac{11}{3} - \left(-\frac{11}{3} \right) = \frac{22}{3}.$$

In some cases, we might not have the function formula for $f(x)$, just a table of data:

Ex. Speedometer readings of a motorcycle are given at 12-second intervals:

sec	t	0	12	24	36	48	60
$\frac{ft}{sec}$	$v(t)$	30	28	25	22	24	27

a) Estimate the distance traveled by the motorcyle during this time period using the velocities at the <u>beginning</u> of the intervals.

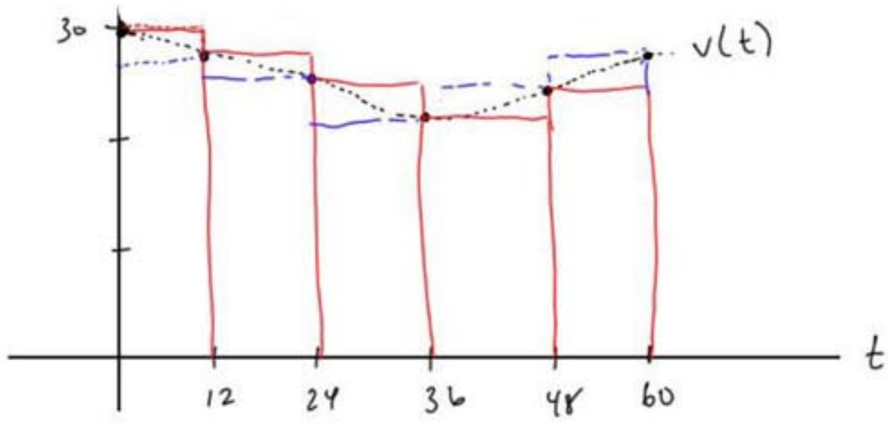
↳ left hand sum.

b) Estimate the distance traveled by the motorcyle during this time period using the velocities at the <u>end</u> of the intervals

↳ right hand sum.

c) Are your estimates upper and lower estimates? Explain.

It may help to plot the data:



a) (left hand sum)
distance traveled $\approx$ $30\frac{ft}{\cancel{s}} \cdot 12\cancel{s} + 28\frac{ft}{s} \cdot 12s + 25\frac{ft}{s} \cdot 12s +$

$+ 22\frac{ft}{\cancel{s}} \cdot 12\cancel{s} + 24\frac{ft}{s} \cdot 12s$

$= 1548$ ft.

$$\begin{array}{r} 129 \\ \times\ 12 \\ \hline 258 \\ 1290 \\ \hline 1548 \end{array}$$

b) (Right hand sum)
distance traveled $\approx 12(28+25+22+24+27)$

$= 1512$ ft.

c) no, not upper or lower estimates because

v(t) decreases then increases.

If $f(x)$ is decreasing only,

left hand sum is an overestimate,

right hand sum is an underestimate.

If $f(x)$ is increasing only,

left hand sum is an underestimate,

right hand sum is an overestimate.