Math 20100
Calculus I

Lesson 22
Antiderivatives

Dr. A. Marchese, The City College of New York

Bookmarks have been added to this video at the following times:

Antiderivatives

Def. $F(x)$ is an antiderivative of $f(x)$

if $F'(x) = f(x)$.

Ex. $F(x) = x^2$ is an antiderivative of $f(x) = 2x$

since $F'(x) = 2x = f(x)$.

But notice, x^2 is not the only antiderivative of $2x$.

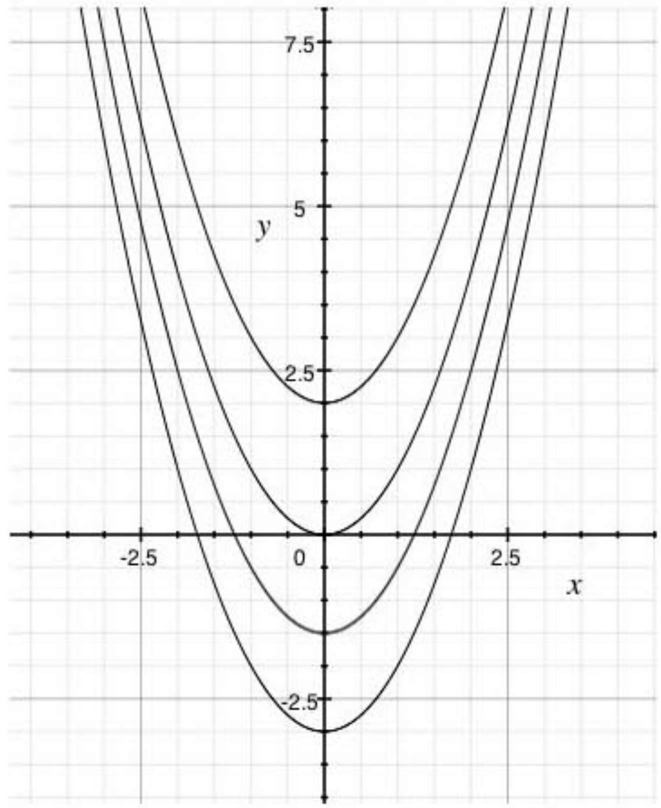
$\frac{d}{dx}(x^2+5) = 2x$, $\frac{d}{dx}(x^2-1) = 2x$, $\frac{d}{dx}(x^2+C) = 2x$

for any constant C.

So we say $F(x) = x^2 + C$ is the (general) antiderivative of $f(x) = 2x$, where C represents any constant.

Notice that for $f(x) = 2x$, the antiderivative $F(x) = x^2 + C$ defines a family of (infinitely many) curves:

Having the same derivative function of $f(x)=2x$ means that at any given x-value, they all have the same slope.



Based on the derivative rules we have learned, we have the following antiderivatives:

$f(x) = 1 \qquad F(x) = x + C$

$f(x) = x^n \qquad F(x) = \dfrac{x^{n+1}}{n+1} + C \quad n \neq -1$

$f(x) = \sin x \qquad F(x) = -\cos x + C$

$f(x) = \cos x \qquad F(x) = \sin x + C$

$f(x) = \sec^2 x \qquad F(x) = \tan x + C$

$f(x) = \sec x \tan x \qquad F(x) = \sec x + C$

Also, if $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively, then the antiderivative of $f(x) \pm g(x)$ is $F(x) \pm G(x) + C$.

And the antiderivative of $kf(x) = kF(x) + C$.

Note: when finding antiderivatives, you can always check your work by taking the derivative of your solution.

Ex. $f(x) = 1 - x^3 + 12x^5$

$$F(x) = x - \frac{x^4}{4} + 12 \cdot \frac{x^6}{6} + C$$

$$= x - \frac{x^4}{4} + 2x^6 + C$$

check: $F'(x) = 1 - \frac{1}{4} \cdot 4x^3 + 2 \cdot 6x^5 + 0$

$$= 1 - x^3 + 12x^5 \checkmark$$

Ex. Find the antiderivative: $f(x) = \frac{2}{x^2} + 5x$

first rewrite $f(x)$: $f(x) = 2x^{-2} + 5x$

then $F(x) = 2 \cdot \frac{x^{-1}}{-1} + 5\frac{x^2}{2} + C$

$= -\frac{2}{x} + \frac{5x^2}{2} + C.$

Ex. Find the antiderivative: $f(x) = \sqrt{x} + \frac{3}{\sqrt{x}}$

first rewrite $f(x)$: $f(x) = x^{1/2} + 3x^{-1/2}$

then $F(x) = \frac{x^{3/2}}{\frac{3}{2}} + 3 \cdot \frac{x^{1/2}}{\frac{1}{2}} + C$

$= \frac{2x^{3/2}}{3} + 6x^{1/2} + C.$

Ex. Find the antiderivative: $f(x) = 2\sin x + \cos x$

$F(x) = 2 \cdot (-\cos x) + \sin x + C$

$= -2\cos x + \sin x + C$

Ex. Find the antiderivative:

$$f(x) = 2 - 6x + 6x^2 - 5\sec^2 x$$

Work on this problem on your own

$$F(x) = 2x - 6\frac{x^2}{2} + 6\frac{x^3}{3} - 5\tan x + C$$

$$= 2x - 3x^2 + 2x^3 - 5\tan x + C.$$

Ex. Find the antiderivative: $f(x) = x^2(2 + \sqrt{x} + x^3)$

We don't yet have a rule for the antiderivative of a product of functions (Calc II), so we have to multiply out first: $f(x) = 2x^2 + x^{5/2} + x^5$

then $F(x) = 2\frac{x^3}{3} + \frac{x^{7/2}}{\frac{7}{2}} + \frac{x^6}{6} + C$

$$= \frac{2}{3}x^3 + \frac{2}{7}x^{7/2} + \frac{1}{6}x^6 + C.$$

Ex. If $f'(x) = 3\sec x \tan x + x$, find $f(x)$.

take the antiderivative:

$$f(x) = 3\sec x + \frac{x^2}{2} + C.$$

Ex. If $f''(x) = -\cos x + 7x^{3/2} + 5$

find $f(x)$.

First we find $f'(x)$: $f'(x) = -\sin x + 7 \cdot \dfrac{x^{5/2}}{5/2} + 5x + C$

$$\therefore \quad f'(x) = -\sin x + \frac{14}{5} x^{5/2} + 5x + C$$

Now take another antiderivative:

another constant

$$f(x) = \cos x + \frac{14}{5} \cdot \frac{x^{7/2}}{\frac{7}{2}} + 5\frac{x^2}{2} + Cx + D$$

$$= \cos x + \frac{\overset{2}{\cancel{14}}}{5} \cdot \frac{2}{\cancel{7}} \cdot x^{7/2} + \frac{5}{2}x^2 + Cx + D$$

$$= \cos x + \frac{4}{5}x^{7/2} + \frac{5}{2}x^2 + Cx + D.$$

Above we were finding general antiderivatives, now let's find particular antiderivatives.

Back to the original example of $f(x) = 2x$ and $F(x) = x^2 + C$.

Ex. Find the (particular) antiderivative of $f(x) = 2x$ with $F(1) = 5$.

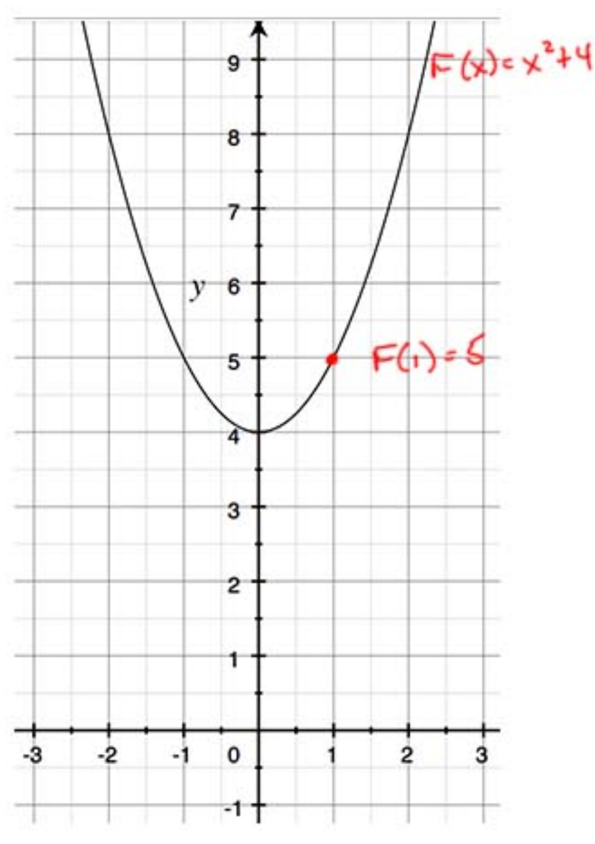
We know any antiderivative of $f(x) = 2x$ is of the form $F(x) = x^2 + C$.

We need $F(1) = 5$.

$$F(1) = 1^2 + C \overset{\text{set}}{=} 5.$$

$$\therefore C = 4$$

$\therefore$ $F(x) = x^2 + 4$ is the (particular) antiderivative of $f(x) = 2x$ with $F(1) = 5$.



Ex. Find $f(x)$. $f'(x) = -2\sin x + \sec^2 x$, $f(\pi) = -1$.

Work on this problem on your own

$$f(x) = -2(-\cos x) + \tan x + C$$

$$= 2\cos x + \tan x + C$$ the general antiderivative

$$f(\pi) = 2\underbrace{\cos\pi}_{-1} + \underbrace{\tan\pi}_{0} + C \overset{\text{set}}{=} -1$$

$$\therefore \quad -2 + C = -1$$

$$+2 \qquad +2$$

$$C = 1$$

$$\therefore \ f(x) = 2\cos x + \tan x + 1.$$ ← be sure to answer the question. must plug in the C you found.

Ex. find $f(x)$. $f''(x) = -\frac{2}{9}x^{-5/3} + 4x^{-7/3}$

with $f'(27) = -\frac{1}{3}$ and $f(27) = 0$

Work on this problem
on your own

first find $f'(x)$: $\quad f'(x) = -\frac{2}{9}\,\frac{x^{-2/3}}{-\frac{2}{3}} + 4\,\frac{x^{-4/3}}{-\frac{4}{3}} + C$

$$= -\frac{2}{9}\cdot\frac{3}{-2}\cdot x^{-2/3} + 4\left(\frac{3}{-4}\right)x^{-4/3} + C$$

$$= \frac{1}{3}x^{-2/3} - 3x^{-4/3} + C.$$

$$= \frac{1}{3x^{2/3}} - \frac{3}{x^{4/3}} + C$$

and $f'(27) = -\frac{1}{3}$ $\quad f'(27) = \frac{1}{3(27)^{2/3}} - \frac{3}{(27)^{4/3}} + C \overset{\text{set}}{=} -\frac{1}{3}$

$$\frac{1}{3\cdot 9} - \frac{3}{81} + C = -\frac{1}{3}$$

$$\frac{1}{27} - \frac{1}{27} + C = -\frac{1}{3}$$

$$\therefore\ C = -\frac{1}{3}$$

$$\therefore\ f'(x) = \frac{1}{3x^{2/3}} - \frac{3}{x^{4/3}} - \frac{1}{3}.$$

$$= \frac{1}{3}x^{-2/3} - 3x^{-4/3} - \frac{1}{3}$$

Now find $f(x)$: $f(x) = \frac{1}{3}\,\frac{x^{1/3}}{1/3} - 3\,\frac{x^{-1/3}}{-\frac{1}{3}} - \frac{1}{3}x + D$

$$= x^{1/3} + 9x^{-1/3} - \frac{1}{3}x + D$$

$$= x^{1/3} + \frac{9}{x^{1/3}} - \frac{1}{3}x + D$$

and $f(27) = 0$ $\quad f(27) = 27^{1/3} + \frac{9}{27^{1/3}} - \frac{1}{3}(27) + D$

$$= 3 + 3 - 9 + D \overset{\text{set}}{=} 0$$

$$-3 + D = 0$$

$$\therefore D = 3$$

$$\therefore f(x) = x^{1/3} + \frac{9}{x^{1/3}} - \frac{1}{3}x + 3.$$

Recall that for a distance function $s(t)$, the velocity $v(t) = s'(t)$, and the acceleration $a(t) = v'(t) = s''(t)$.

Using the methods above, for objects moving with constant acceleration, if we know the initial velocity and position, we can find the distance s at any time.

Ex. A penny is dropped from the observation deck of the Empire State Building (1250 ft above ground).

a) find the height of the penny (above ground level) at any time t.

b) What is the velocity of the penny when it hits the ground?

c) if the penny was thrown downward with a velocity of 16 ft/sec, how long would it take to reach the ground?

a) we know that gravity (neglecting air resistance) provides a constant acceleration on the penny of $-32\ ft/s^2$, and this is the only force governing the movement.

$\therefore\ a(t) = -32 \quad \Rightarrow \quad v(t) = -32t + C$ ← constant

& we know $v(0) = 0$ since the penny was dropped.

$$V(0) = -32(0) + C \overset{set}{=} 0 \quad \Rightarrow C = 0$$

$\therefore\ v(t) = -32t \quad \Rightarrow \quad s(t) = -32\frac{t^2}{2} + D$ ← constant

$$= -16t^2 + D$$

& we know $S(0) = 1250$ = height of observation deck.

$$S(0) = -16 \cdot 0^2 + D \overset{\text{set}}{=} 1250$$

$$\therefore D = 1250$$

& $s(t) = -16t^2 + 1250$. ← answer to part a.

b) first we need to know when (what time) the penny hits the ground, ie at what time $s(t) = 0$.

$$s(t) = -16t^2 + 1250 \overset{\text{set}}{=} 0$$

$$\therefore t^2 = \frac{1250}{16} = \frac{625}{8}$$

$$\& \quad t = \sqrt{\frac{625}{8}} = \frac{25}{2\sqrt{2}} \approx 8.8 \text{ sec}$$

$$v\left(\frac{25}{2\sqrt{2}}\right) = -32\left(\frac{25}{2\sqrt{2}}\right) = -\frac{400}{\sqrt{2}} = -\frac{400\sqrt{2}}{2} = -200\sqrt{2} \text{ ft/s}$$

$$\approx -282.8 \text{ ft/s}$$

c) if $v(0) = -16$ ft/s, then $v(t) = -32t - 16$

↑

thrown

downward, in direction of decreasing height and $s(t) = -16t^2 - 16t + D_2$

$$s(0)=1250 \Rightarrow \quad 1250 = -16(0)^2 - 16(0) + D_2$$
$$\Rightarrow D_2 = 1250$$

$$s(t) = -16t^2 - 16t + 1250.$$

The penny would hit the ground when $s(t) = 0$

$$-16t^2 - 16t + 1250 = 0$$

$$t = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(-16)(1250)}}{2(-16)} = \frac{16 \pm \sqrt{256 + 80000}}{-32}$$

$$= \frac{16 \pm \sqrt{80256}}{-32} = \frac{16 \pm 8\sqrt{1254}}{-32}$$

need a positive t value, so $t = \dfrac{16 - 8\sqrt{1254}}{-32} \approx 8.4$ s.

should be faster than the
time above, since it starts with
a downward velocity
(and it is)

Ex. A car is traveling at 60 mi/h when the brakes are applied, giving a constant deceleration of 15 ft/s^2. How far does the car travel before coming to a stop?

Work on this problem on your own

We are asked to find $s(t)$ at the time when $v(t) = 0$.

Given $a(t) = -15$ ft/s^2

$$\Rightarrow \quad v(t) = -15t + C \quad ft/s$$

We are given $v(0) = 60$ mi/h need this in ft/s

$$\cancel{60}\ \frac{\cancel{mi}}{\cancel{h}} \cdot \frac{5280\ ft}{\cancel{mi}} \cdot \frac{1\ \cancel{h}}{\underset{60}{\cancel{3600}}\ s} = \frac{5280}{60}\ \frac{ft}{s} = 88\ ft/s.$$

So $v(0) = 88$ ft/s $\quad \Rightarrow \quad -15(0) + C = 88$

$$\therefore C = 88$$

and $v(t) = -15t + 88$ ft/s.

so $V(t) = 0$: $0 = -15t + 88$

$$t = \frac{88}{15} \text{ s} \approx 5.9 \text{ s}$$

$$S(t) = -15\frac{t^2}{2} + 88t + D$$

we can let $s(0) = 0$ and measure distance from the point at which the brakes were applied

then $s(0) = -15\frac{(0)^2}{2} + 88(0) + D = 0$

$$\therefore D = 0$$

and $S(t) = -15\frac{t^2}{2} + 88t$. this gives how far the car has traveled t sec after brakes are applied

$$S\left(\frac{88}{15}\right) = -\frac{15}{2}\left(\frac{88}{15}\right)^2 + 88\left(\frac{88}{15}\right) =$$

$$= \frac{-\cancel{15}}{2} \cdot \frac{88}{\cancel{15}} \cdot \frac{88}{15} + \frac{88^2}{15} = \frac{1}{2}\left(\frac{88^2}{15}\right) =$$

$$= \frac{7744}{30} \text{ ft} \approx 258.1 \text{ ft.}$$