Math 20100
Calculus I

Lesson 21
Optimization

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Optimization

(Applied Maximum/Minimum Problems)

Ex. A box with a square base and open top must have a volume of 4000 cm^3. Find the dimensions of the box so that the amount of material is minimized.

Guidelines for solving Optimization Problems:

1. make a sketch (if necessary) label (with variables) the quantities you are trying to find.
2. Write down the function to be optimized (maximized or minimized). This is your primary equation.

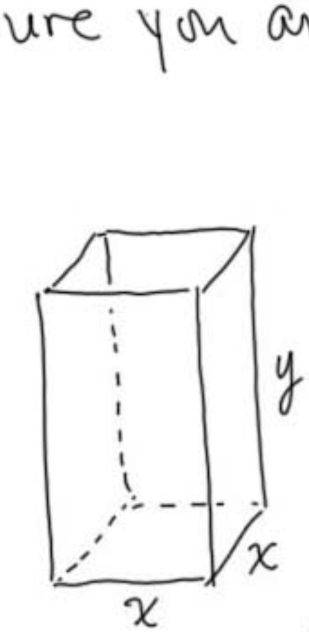
③ If your primary equation is a function of two variables, find a secondary equation relating the two variables. Use substitution to get the Primary equation as a function of one variable.

④ Use calculus to find your critical numbers

⑤ Test critical numbers to see if you have max/min. (use the Second Derivative Test for Relative Extrema)

⑥ Make sure you answer the question

Ex above:



We need to find x and y such that surface area is minimized

bottom　　sides

Surface Area $S = x^2 + 4xy$ (open top)

← Primary Equation

Secondary equation: volume = 4000 cm^3

$$V = x^2 y \Rightarrow x^2 y = 4000$$

Use substitution to get primary equation in one variable: $y = \frac{4000}{x^2}$

$$\therefore\ S = x^2 + 4x\left(\frac{4000}{x^2}\right) = x^2 + \frac{16000}{x}$$

Find critical numbers: $S = x^2 + 16000x^{-1}$

$$S' = 2x - 16000x^{-2} = 2x - \frac{16000}{x^2}$$

S' DNE at $x = 0$ (if $x = 0$, no box)

$S' = 0$ when $2x - \frac{16000}{x^2} = 0$

$$2x = \frac{16000}{x^2}$$

$$2x^3 = 16000$$

$$x^3 = 8000 \Rightarrow x = 20$$

To verify that $x = 20$ gives a minimum, use the Second Derivative Test:

$$S'' = 2 + 32000x^{-3} = 2 + \frac{32000}{x^3}$$

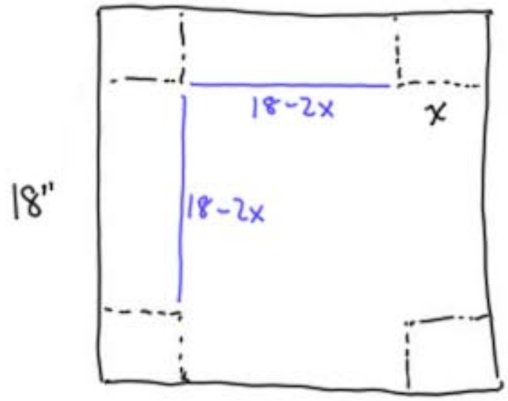
$$S''(20) = 2 + \frac{32000}{(20)^3} > 0 \quad \therefore \ x = 20 \text{ gives a minimum.}$$

$x = 20$ is the only critical # for $x > 0$, must yield the absolute min.

$$\therefore \ x = 20 \text{ and } y = \frac{4000}{x^2} = \frac{4000}{(20)^2} = 10$$

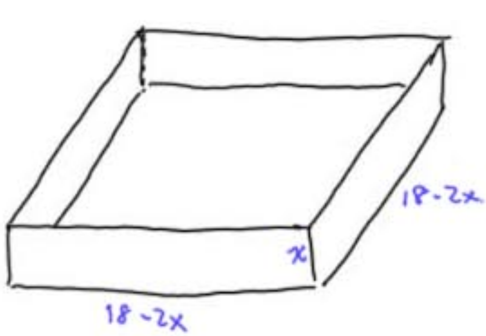
$\therefore$ The box should have a base of 20 cm $\times$ 20 cm and a height of 10 cm.

Ex. A box with an open top is to be constructed from a square piece of cardboard, 18 in wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest possible volume.



Note that what can vary here is the size of the corner square, give that side length a variable name, x.

⇓



Then we need a formula for Volume as a function of x.

$$V = x(18-2x)^2 \quad \text{primary equation}$$

$$V' = \underbrace{\frac{d}{dx}(x)}_{1}(18-2x)^2 + x \cdot \frac{d}{dx}\left((18-2x)^2\right)$$

$$V' = (18-2x)^2 + x \cdot 2(18-2x)(-2)$$

always exists, set $= 0$

$$(18-2x)^2 - 4x(18-2x) = 0$$

$$(18-2x)\left[18-2x-4x\right] = 0$$

$$18-2x = 0 \qquad 18-6x = 0$$

$$18 = 2x \qquad 18 = 6x$$

$$\cancel{x=9} \text{ or } \quad x = 3$$

no box

Check that $x = 3$ gives a maximum for Volume.

We know $V'(3) = 0$, check sign of $V''(3)$.

$$V' = (18-2x)^2 - 4x(18-2x)$$

$$V'' = 2(18-2x)(-2) - \left(4(18-2x) + 4x(-2)\right)$$

$$= -4(18-2x) - 4(18-2x) + 8x$$

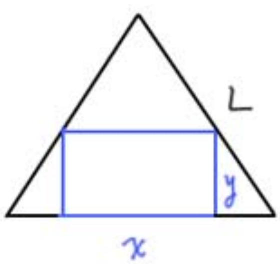
$$= -8(18-2x) + 8x$$

$V''(3) = -8(18-6) + 8(3) < 0 \Rightarrow$ maximum at $x = 3$

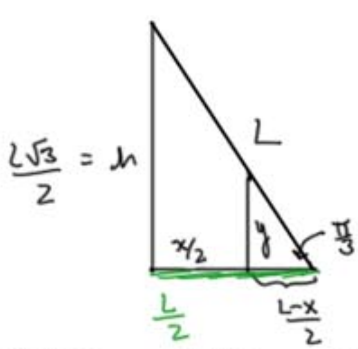
absolute max since it is the only critical # in our domain $(0,9)$

The maximum volume possible is $V = 3(18-2(3))^2 = 3(12)^2$
$= 432 \text{ in}^3$

Ex. Find the dimensions of the rectangle of the largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.



$A = xy$ primary equation



height of the original triangle:

$$\sin\frac{\pi}{3} = \frac{h}{L}$$

$$\frac{\sqrt{3}}{2} = \frac{h}{L} \Rightarrow h = \frac{L\sqrt{3}}{2}$$

$$\frac{2}{2}\cdot\frac{\frac{L\sqrt{3}}{2}}{\frac{L}{2}} = \frac{y}{\frac{L-x}{2}}\cdot\frac{2}{2}$$

$$\frac{\cancel{L}\sqrt{3}}{\cancel{L}} = \frac{2y}{L-x}$$

$$\sqrt{3}(L-x) = 2y$$

$$y = \frac{\sqrt{3}(L-x)}{2}$$

$$A = xy = x\cdot\frac{\sqrt{3}(L-x)}{2} = \frac{L\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}x^2$$

$$A' = \frac{L\sqrt{3}}{2} - \frac{\sqrt{3}}{\cancel{2}}\cancel{2}x = \frac{L\sqrt{3}}{2} - \sqrt{3}x$$ always exists

$$\frac{L\sqrt{3}}{2} - \sqrt{3}x = 0 \qquad \frac{\frac{L\sqrt{3}}{2}}{\sqrt{3}} = \frac{\sqrt{3}x}{\sqrt{3}}$$

$$x = \frac{L}{2}$$ check that $x = \frac{L}{2}$ gives max area

$A'' = -\sqrt{3} < 0 \Rightarrow A$ concave down, max at $x = \frac{L}{2}$
absolute max since A is downward facing parabola.

$$x = \frac{L}{2} \quad y = \frac{\sqrt{3}(L-x)}{2} = \frac{\sqrt{3}\left(L-\frac{L}{2}\right)}{2} = \frac{\sqrt{3}\left(\frac{L}{2}\right)}{2} = \frac{\sqrt{3}}{4}L$$

dimensions of the rectangle of largest area:

$\frac{L}{2} \times \frac{\sqrt{3}}{4}L$.

Ex. At which points on The curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have The largest slope?

No sketch necessary. $y' = 120x^2 - 15x^4$ ← primary equation.
$= S$ (slope)

Need to maximize S.

$S' = y'' = 240x - 60x^3 = 0$ (always exists)

$60x(4 - x^2) = 0$

$60x = 0 \qquad 4 - x^2 = 0$

$x = 0 \qquad x = \pm 2.$

$S'' = y''' = 240 - 180x^2$

at $x = 0$, $y''' = 240 > 0 \Rightarrow$ minimum of S
S concave up

at $x = \pm 2$, $y''' = 240 - 180(4) < 0 \Rightarrow$ maximum of S.
S concave down

We have 2 relative maximums for The slope function. Is one slope larger Than The other?

notice $S = y' = 120x^2 - 15x^4$ ← only even powers

Same for $x = \pm 2$

$S = 120(4) - 15(16) = 240$

need The y-coordinates : $y = 1 + 40x^3 - 3x^5$

$x = -2, \quad y = 1 + 40(-8) - 3(-32) = 1 - 320 + 96 = -223$

$x = 2, \quad y = 1 + 40(8) - 3(32) = 1 + 320 - 96 = 225$

$(-2, -223)$ + $(2, 225)$.