Math 20100
Calculus I

Lesson 20
Curve Sketching

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Curve Sketching

So far we've learned that

$$f'(x) > 0 \Rightarrow f \text{ is increasing}$$

$$f'(x) < 0 \Rightarrow f \text{ is decreasing}$$

$$f''(x) > 0 \Rightarrow f' \text{ is increasing} \Rightarrow f \text{ is concave up}$$

$$f''(x) < 0 \Rightarrow f' \text{ is decreasing} \Rightarrow f \text{ is concave down}$$

In this lesson, we use this information to help us sketch the graphs of functions.

Steps for curve sketching

1. find the domain
2. find and plot any intercepts and asymptotes (vertical and horizontal)
3. find and plot any relative extrema, and find intervals of increase/decrease

④ find and plot inflection points, and find intervals of concavity

⑤ sketch!

Ex. Use calculus to sketch the graph of

$y = x^4 - 6x^2 + 8$

$y(\pm\sqrt{3}) = 9 - 6(3) + 8 = -1$

$y(\pm 1) = 1 - 6 + 8 = 3$

① domain $\mathbb{R}$

② intercepts: y-int $x = 0$: $y = 0^4 - 6 \cdot 0^2 + 8 = 8$

$(0, 8)$

x-int $y = 0$: $0 = x^4 - 6x^2 + 8$

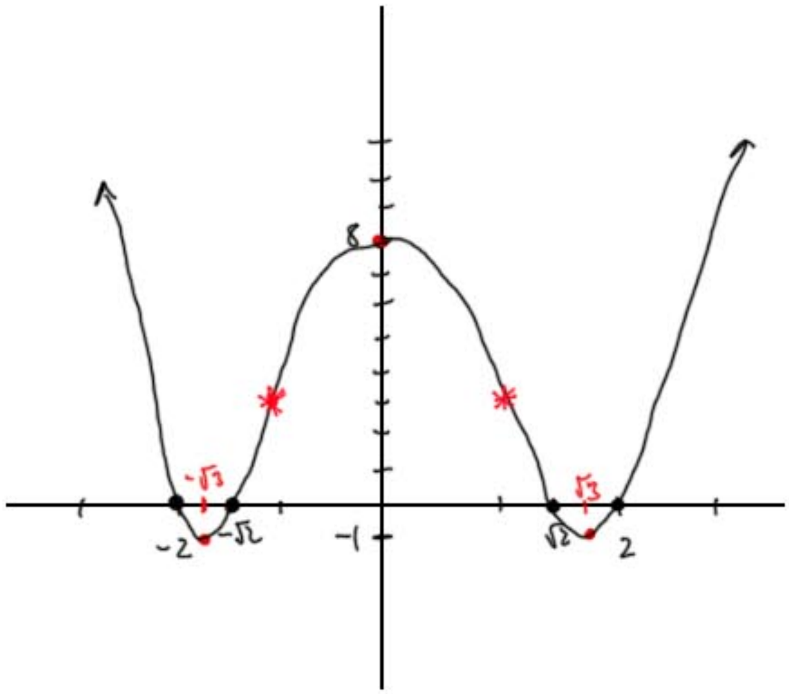
$(x^2 - 4)(x^2 - 2)$

$x^2 - 4 = 0 \quad x^2 - 2 = 0$

$x = \pm 2 \quad x = \pm\sqrt{2}$

$(\pm\sqrt{2}, 0) \quad (\pm 2, 0)$

no asymptotes for polynomials



③ relative extrema

$$y = x^4 - 6x^2 + 8$$

$$y' = 4x^3 - 12x \overset{\text{set}}{=} 0 \qquad \text{(always exists)}$$

$$= 4x(x^2 - 3) = 0$$

$$x = 0 \qquad x = \pm\sqrt{3}$$

y: dec ∪ inc ∩ dec ∪ inc

y': − min + max − min +

$-\sqrt{3}$ 0 $\sqrt{3}$

$x=-2$ $x=-1$ $x=1$ $x=2$

relative min
$(-\sqrt{3}, -1)$
$(\sqrt{3}, -1)$

relative max
$(0, 8)$

$$y' = 4x(x^2-3) \qquad y'(-2) = -\underbrace{(4-3)}_{+} = - \qquad y'(-1) = -\underbrace{(1-3)}_{-} = +$$

$y'(1) = +(-) = -$ $\quad$ $y'(2) = +(+) = +$

y increasing on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

y decreasing on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

④ inflection + concavity

$y'' = 12x^2 - 12$ $\quad$ (always exists)

$= 12(x^2 - 1) = 0$

$x = \pm 1$

y	conc up		conc down		conc up
y'	inc		dec		inc
y''	$+$		$-$		$+$
	$x=-2$	-1	$x=0$	1	$x=2$

$x = \pm 1$ points of inflection

$(-1, 3)$
$(1, 3)$

$y'' = 12(x^2 - 1)$

$y''(-2) = +\underbrace{(4-1)}_{+} = +$

$y''(0) = +(-1) = -$

$y''(2) = +\underbrace{(4-3)}_{+} = +$

Ex. Use calculus to sketch the graph of $f(x) = \frac{x^2}{x^2-1}$

① domain: $x^2 - 1 \neq 0$ so $x \neq \pm 1$

② intercepts: y-int, $x=0$ $y = \frac{0}{0-1} = 0$ $(0,0)$

x-int, $y=0$ $0 = \frac{x^2}{x^2-1} \Rightarrow x^2 = 0$
$x = 0$

asymptotes: check vertical asymptotes at $x = \pm 1$

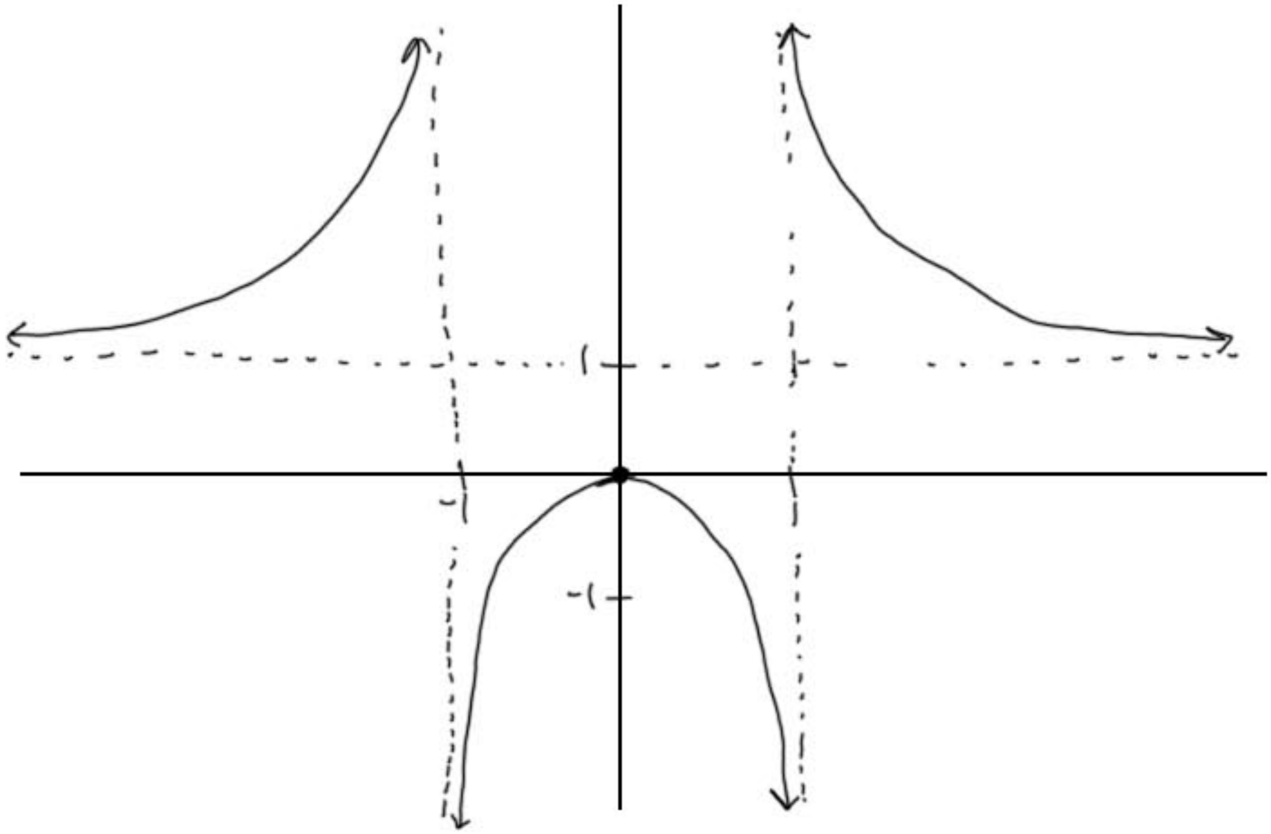
$f(1) = \frac{1^2}{1^2-1} = \frac{1}{0}$ $\frac{\text{nonzero}}{\text{zero}}$ ✓

$f(-1) = \frac{(-1)^2}{1-1} = \frac{\text{nonzero}}{\text{zero}}$ ✓

horizontal asymptotes:

$$\lim_{\substack{x \to \infty \\ x \to -\infty}} \frac{x^2}{x^2-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to \pm\infty} \frac{1}{1 - \frac{1}{x^2}} = \frac{1}{1-0} = 1$$

$\therefore$ $y=1$ is a horizontal asymptote as $x \to \infty$
and $x \to -\infty$.

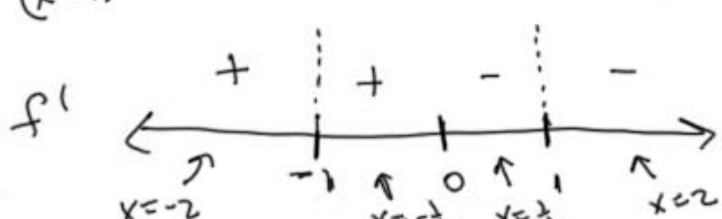


③ $f'(x) = \dfrac{(x^2-1)(2x) - (x^2)(2x)}{(x^2-1)^2} = \dfrac{2x[\cancel{x^2}-1-\cancel{x^2}]}{(x^2-1)^2}$

$= \dfrac{-2x}{(x^2-1)^2}$ DNE at $x = \pm 1$
asymptotes of f
(not critical numbers)

$f'(x) = 0$ when $-2x=0$, $x=0$ critical number

$f'(x) = \dfrac{-2x}{(x^2-1)^2} \leftarrow +$



Can make a basic sketch already,
then make sure concavity agrees.

(4)

$$f''(x) = \frac{(x^2-1)^2(-2) - (-2x)\,2(x^2-1)(2x)}{(x^2-1)^4}$$

$$= \frac{\cancel{(x^2-1)}\left[(x^2-1)(-2) + 8x^2\right]}{(x^2-1)^{\cancel{4}\,3}} = \frac{-2x^2+2+8x^2}{(x^2-1)^3}$$

$$f''(x) = \frac{6x^2+2}{(x^2-1)^3}$$ DNE at $x = \pm 1$ asymptotes

$6x^2+2=0$

$6x^2=-2$

no solution

f	conc up	conc dwn	conc up
f'	inc	dec	inc
f''	+	−	+

-1 $\quad$ 1

$x=-2$ $\quad$ $x=0$ $\quad$ $x=2$

$f''(-2)$ $\quad \frac{6(-2)^2+2}{(4-1)^3}$ $\quad \frac{+}{+}$

$f''(0) = \frac{2}{(-1)^3} = -$

$f''(2) = \frac{6(4)+2}{(4-1)^3} = +$

Ex. $f(x) = \sqrt{x^2+1} - x$ Use calculus to sketch

① domain: need $x^2+1 \geq 0$

always true since $x^2 \geq 0$

$\therefore$ domain $\mathbb{R}$

(no vertical asymptotes)

② intercepts y-int, $x=0$ $f(0) = \sqrt{1} - 0 = 1$

$(0,1)$

x-int, $y=0$

$$0 = \sqrt{x^2+1} - x$$
$$+x \qquad\qquad +x$$

$$x = \sqrt{x^2+1}$$

$$x^2 = x^2+1$$
$$-x^2 \quad -x^2$$
$$0 = 1$$

no solution, no x-intercepts.

asymptotes domain $\mathbb{R}$ no vertical asymp.

horizontal

$$\lim_{x \to \infty} \sqrt{x^2+1} - x$$

$$\infty - \infty$$

an indeterminate form

$\Rightarrow$ we need more info

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) \frac{\left(\sqrt{x^2+1} + x\right)}{\left(\sqrt{x^2+1} + x\right)} =$$

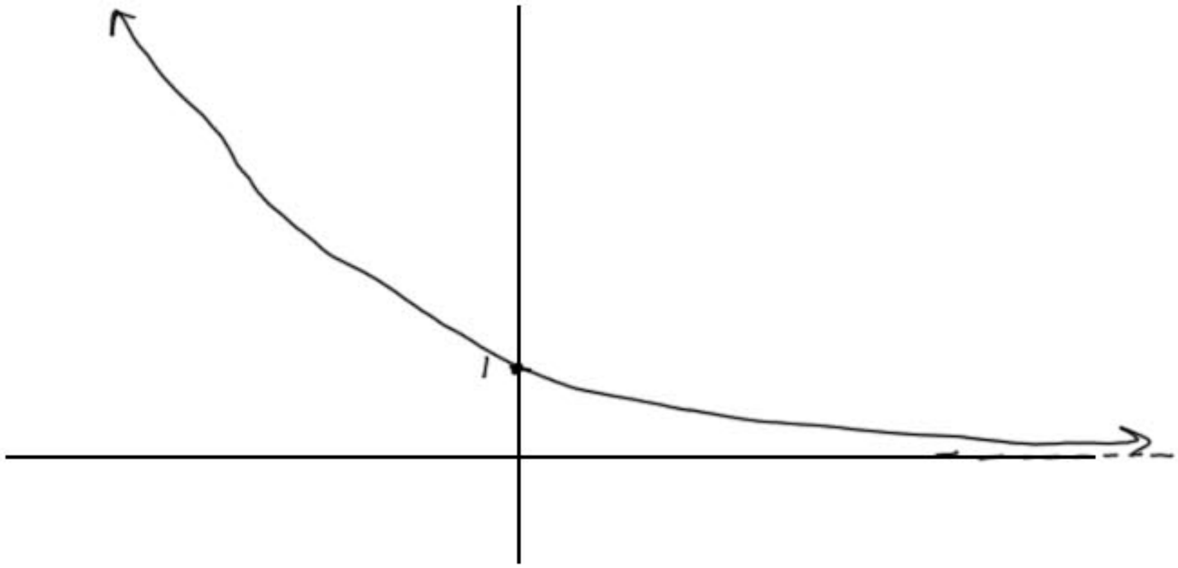
$$= \lim_{x \to \infty} \frac{x^2+1-x^2}{\sqrt{x^2+1} + x} = \lim_{x \to \infty} \frac{1}{\underbrace{\sqrt{x^2+1} + x}_{\underbrace{\infty + \infty}_{\infty}}} = 0$$

$\therefore$ $y = 0$ is a horizontal asymptote as $x \to \infty$

now $\lim_{x \to -\infty} \sqrt{x^2+1} - x$

$$\infty - (-\infty) = \infty + \infty = \infty$$

no horizontal asymptote as $x \to -\infty$



③ $f'(x) = \frac{1}{2}(x^2+1)^{-1/2}(2x) - 1$

$$= \frac{x}{\sqrt{x^2+1}} - 1$$

always exists

$$\frac{x}{\sqrt{x^2+1}} - 1 = 0$$

$$\frac{x}{\sqrt{x^2+1}} = 1$$

$x = \sqrt{x^2+1}$

$x^2 = x^2+1$ no solution

no critical numbers

f' $\quad -$

$\longleftrightarrow$

$\uparrow$

test $x=0$ (or any x-value)

$\therefore$ f is always decreasing

④ $f'(x) = \dfrac{x}{\sqrt{x^2+1}} - 1$

$$f''(x) = \frac{\sqrt{x^2+1}\,(1) - x \cdot \cancel{\tfrac{1}{2}}(x^2+1)^{-1/2}\,(\cancel{2}x)}{\left(\sqrt{x^2+1}\right)^2} \cdot \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$= \frac{x^2+1 - x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}} \neq 0 \quad \text{always exists}$$

f'' $\quad$ +

$\longleftrightarrow$

test $x=0$ (or any x-value)

f is always concave up.

Ex. Use calculus to sketch the graph of

$y = 2x - \tan x$ on $(-\pi/2, \pi/2)$

① domain is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (given)

② y-int: $x = 0$ $\quad y = 2(0) - \tan(0) = 0 \quad (0,0)$

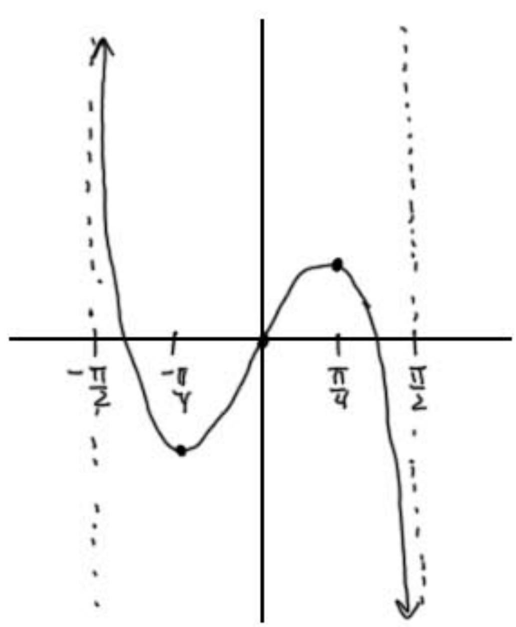
x-int: $y = 0 \quad 0 = 2x - \tan x$

$2x = \tan x$ (?)

don't worry about it, move on

asymptotes: no horizontal since we have a restricted domain

vertical when $\cos x = 0$, ie $x = \pm\frac{\pi}{2}$



③ relative extrema $\quad y = 2x - \tan x$

$y' = 2 - \sec^2 x \overset{\text{set}}{=} 0$

$y\left(-\frac{\pi}{4}\right) = 2\left(-\frac{\pi}{4}\right) - \underbrace{\tan\left(-\frac{\pi}{4}\right)}_{-1}$

$-\frac{\pi}{2} + 1$

$\approx -\frac{1}{2}$

also note $2 - \sec^2 x$ DNE

where $\cos x = 0$,

same as original

$x = \pm\frac{\pi}{2}$

$2 = \sec^2 x$

$2 = \frac{1}{\cos^2 x}$

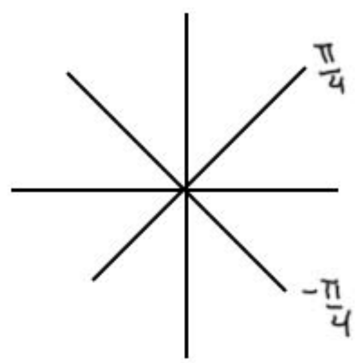
$\cos^2 x = \frac{1}{2}$

$\cos x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$

$y\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)$

$\frac{\pi}{2} - 1 \approx \frac{1}{2}$

reference angle $\frac{\pi}{4}$, in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ QI + IV



$x = \pm\frac{\pi}{4}$.

critical numbers

relative min $\left(-\frac{\pi}{4}, -\frac{\pi}{2}+1\right)$

relative max $\left(\frac{\pi}{4}, \frac{\pi}{2}-1\right)$

$y' = 2 - \sec^2 x$

y dec ∪ inc ∩ dec

y' − min + max −

$-\frac{\pi}{2}$ $\quad -\frac{\pi}{4}$ $\quad \frac{\pi}{4}$ $\quad \frac{\pi}{2}$

$x = -\frac{\pi}{3}$ $\quad x = 0$ $\quad x = \frac{\pi}{3}$

$y'\left(-\frac{\pi}{3}\right) =$

$2 - \sec^2\left(-\frac{\pi}{3}\right)$

$2 - \left(\frac{1}{\cos\left(-\frac{\pi}{3}\right)}\right)^2 = -$

$y'(0) = 2 - \sec^2(0) = +$

$y'\left(\frac{\pi}{3}\right) = 2 - \sec^2\left(\frac{\pi}{3}\right)$

$2 - \left(\frac{1}{\frac{1}{2}}\right)^2$

$= -$

④ inflection + concavity

$$y' = 2 - \sec^2 x = 2 - (\sec x)^2$$

$$y'' = -2(\sec x) \cdot \frac{d}{dx}(\sec x)$$

$$= -2 \sec x \sec x \tan x$$

$$= -2 \sec^2 x \tan x \quad \text{DNE at } x = \pm \frac{\pi}{2}$$

$$0 = -2 \sec^2 x \tan x$$

$\sec^2 x \neq 0$

$\frac{1}{\cos^2 x} \neq 0$

$\tan x = 0$

$\Rightarrow \sin x = 0 \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$x = 0.$

y	conc up	conc dwn.
y'	inc	dec
y''	$+$	$-$

$\left(-\frac{\pi}{2}\right.$ ——— 0 ——— $\left.\frac{\pi}{2}\right)$

$x = -\frac{\pi}{4}$ $\quad$ $x = \frac{\pi}{4}$

inflection point at $x = 0$ $(0, 0)$

$$y'' = -2 \sec^2 x \tan x$$
$$-(+) \tan x$$

$$y''\left(-\frac{\pi}{4}\right) = -(+)(-1) = +$$

$$y''\left(\frac{\pi}{4}\right) = -(+)(+1) = -$$