# Math 20100
# Calculus I

# Lesson 19
# Concavity

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# Concavity

In this lesson we continue to examine what derivatives tell us about the shape of the graph of a function.

Definitions:

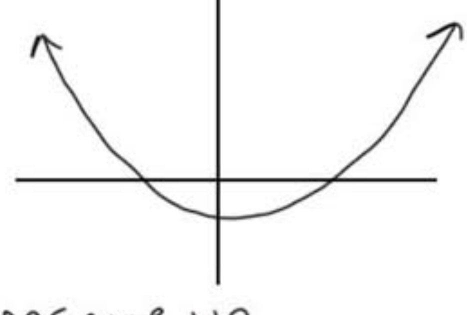
$f$ is concave up where $f'$ is increasing

(the slope of $f$ is increasing)

$f$ is concave down where $f'$ is decreasing

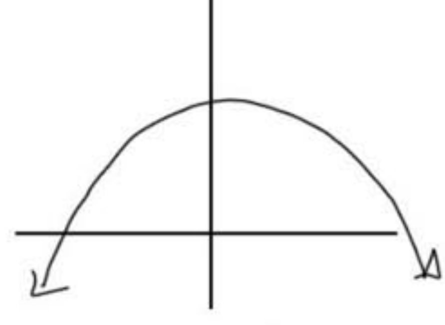
(the slope of $f$ is decreasing)

Ex.



concave up

cup



concave down

frown

To see where a function ($f'$) is increasing or decreasing, we look at the sign of its derivative ($f''$).

So the sign of $f''$ will give the concavity of $f$.

$f$ is concave up where $f'' > 0$, $f'$ is increasing

$f$ is concave down where $f'' < 0$. $f'$ is decreasing

<u>Definition:</u> If $f$ changes concavity at $x = a$, $(a, f(a))$ is called an <u>inflection point</u> of $f$.

Let's take a look at an example from Lesson 18, $f(x) = x^4 - 4x^2 - 1$. We'll find the intervals on which $f$ is concave up and down, and inflection points.

# To find intervals of concavity and inflection points:

1) find any discontinuities of $f$
   (check the domain)

2) find $f''(x)$

3) find the x-values for which $f''(x)=0$
   or $f''(x)$ DNE

4) plot these numbers and discontinuities
   on a number line

5) find the sign of $f''$ on each interval of
   the number line

6) $f$ is concave up where $f''>0$,

   $f$ is concave down where $f''<0$.

   If $f$ changes concavity at $x=a$,
   $(a, f(a))$ is an inflection point of $f$.

Ex. from Lesson 18 $\quad f(x) = x^4 - 4x^2 - 1 \quad$ domain $\mathbb{R}$

$$f'(x) = 4x^3 - 8x$$

$$f''(x) = 12x^2 - 8 \qquad \text{always exists}$$

$$12x^2 - 8 = 0$$

$$12x^2 = 8$$

$$x^2 = \frac{8}{12} = \frac{2}{3}$$

$$x = \pm\sqrt{\frac{2}{3}}$$

| | $x=-1$ | $-\sqrt{\frac{2}{3}}$ | $x=0$ | $+\sqrt{\frac{2}{3}}$ | $x=1$ |
|---|---|---|---|---|---|
| $f$ | conc. up | * | conc. down | * | conc. up. |
| $f'$ | inc | | dec | | inc |
| $f''$ | + | | − | | + |

inflection points $x = \pm\sqrt{\frac{2}{3}}$ since concavity changes at each of them

$f$ concave up on $\left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

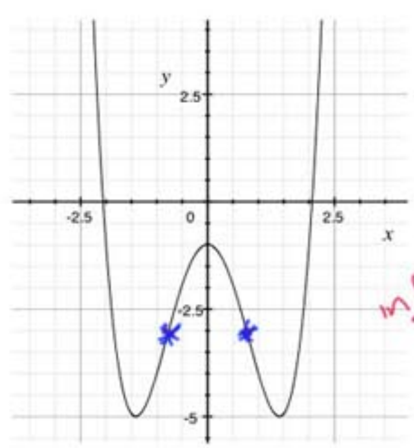
$f$ concave down on $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

$f''(x) = 12x^2 - 8$

$f''(-1) = 12(-1)^2 - 8 = +$

$f''(0) = -8$

$f''(1) = 12(1)^2 - 8 = +$




inflection points

Ex. from lesson 18, $f(x) = \frac{x^2}{x+1}$ domain: $(-\infty,-1)\cup(-1,\infty)$

we found $f'(x) = \frac{x^2+2x}{(x+1)^2}$

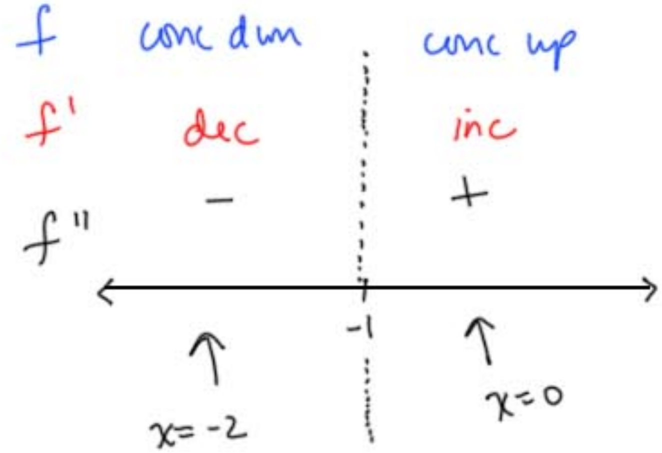
then $$f''(x) = \frac{(x+1)^2(2x+2) - (x^2+2x)2(x+1)(1)}{(x+1)^4}$$

$$= \frac{\cancel{(x+1)}\left[(x+1)(2x+2) - (x^2+2x)(2)\right]}{(x+1)^{\cancel{4}3}}$$

$$= \frac{2x^2+2x+2x+2-2x^2-4x}{(x+1)^3}$$

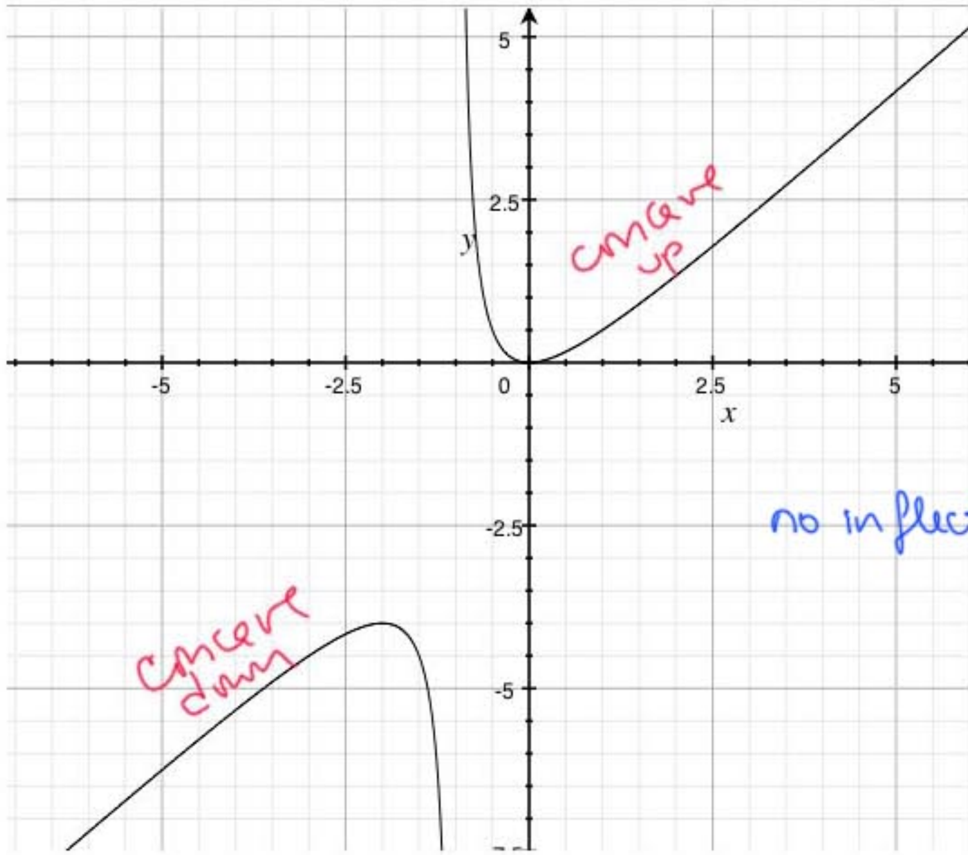
$$= \frac{2}{(x+1)^3}$$ DNE at $x=-1$, not in domain of $f$ (asymptote)

never $=0$




$f''(-2) = \frac{2}{(-2+1)^3} = \frac{2}{(-1)^3} = -$

$f''(0) = \frac{2}{(0+1)^3} = +$




Ex. Find the intervals of concavity and inflection points: $f(x) = \cos^2 x - 2\sin x$

for $0 \le x \le 2\pi$

$$f'(x) = 2\cos x(-\sin x) - 2\cos x$$

$$= -2\sin x \cos x - 2\cos x$$

$$f''(x) = -2(\cos x \cos x + \sin x(-\sin x)) - 2(-\sin x)$$

$$= -2\cos^2 x + 2\sin^2 x + 2\sin x$$

exists $\forall x \in [0, 2\pi]$, need to set $= 0$.

$$-2\cos^2 x + 2\sin^2 x + 2\sin x = 0.$$

In this case, best to replace the $\cos^2 x$ with $1 - \sin^2 x$ so we have everything in terms of $\sin x$.

$$-2(1 - \sin^2 x) + 2\sin^2 x + 2\sin x = 0$$

$$-2 + 2\sin^2 x + 2\sin^2 x + 2\sin x = 0$$

$$4\sin^2 x + 2\sin x - 2 = 0 \qquad \div 2$$

$$2\sin^2 x + \sin x - 1 = 0$$ quadratic, factor

$$(2\sin x - 1)(\sin x + 1) = 0$$

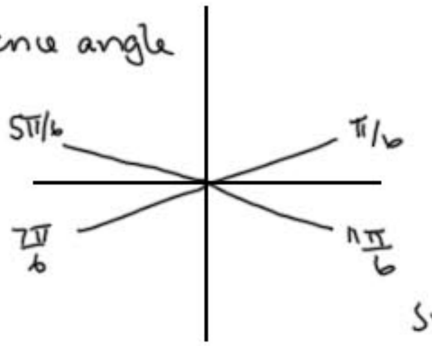
$$2\sin x - 1 = 0 \qquad \sin x + 1 = 0$$

| | |
|---|---|
| $\sin x = \frac{1}{2}$ | $\sin x = -1$ |
| we know $x = \frac{\pi}{6}$ is | $x = \frac{3\pi}{2}$ |

need solutions in $[0, 2\pi]$

a solution, any others in other quadrants?

$\frac{\pi}{6}$ reference angle




Sine is positive in QI + QII

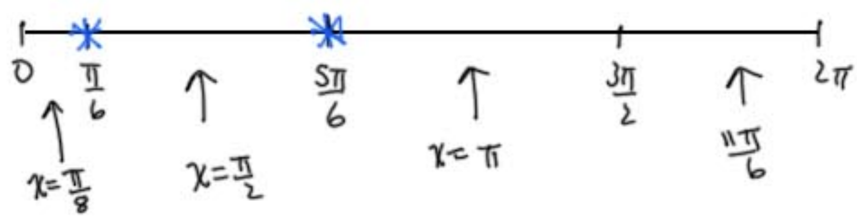
So $x = \frac{5\pi}{6}$ is also a solution

$\sin\frac{11\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

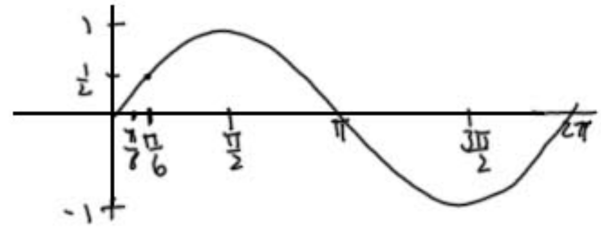
$f''(x) = 0$ at $x = \frac{\pi}{6}$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{6}$

| | | | | |
|---|---|---|---|---|
| $f$ | conc dwn | conc up | conc dwn | conc dwn |
| $f'$ | dec | inc | dec | dec |
| $f''$ | $-$ | $+$ | $-$ | $-$ |

inflection points at $x = \frac{\pi}{6}, \frac{5\pi}{6}$.




$f''(x) = 2(2\sin x - 1)(\sin x + 1)$

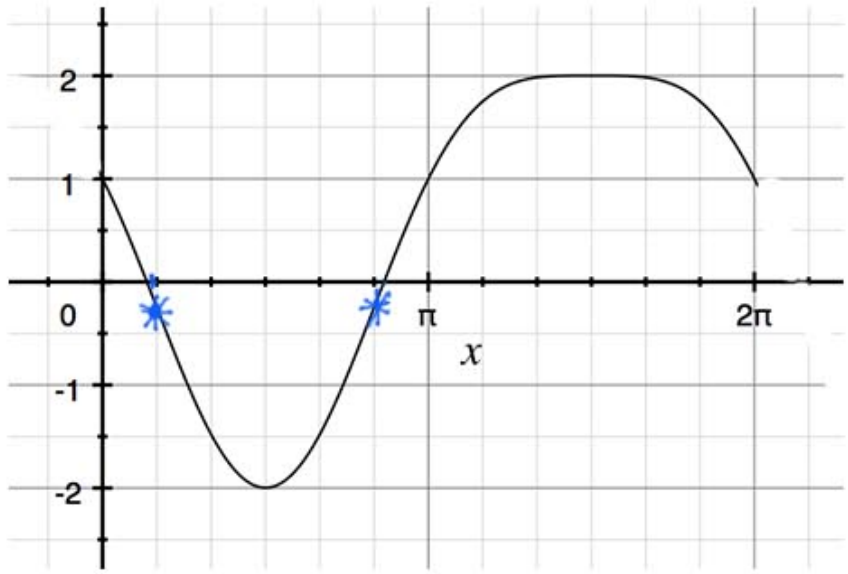



$f''(\frac{\pi}{8}) = 2(\;-\;)\;+\; = -$

$f''(\frac{\pi}{2}) = 2(2(1)-1)(1+1) = +$

$f''(\pi) = 2(0-1)(0+1) = -$

$f''(\frac{11\pi}{6}) = 2(2(-\frac{1}{2})-1)(-\frac{1}{2}+1) = -$
$2(-)(+)$




f is concave up on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

f is concave down on $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

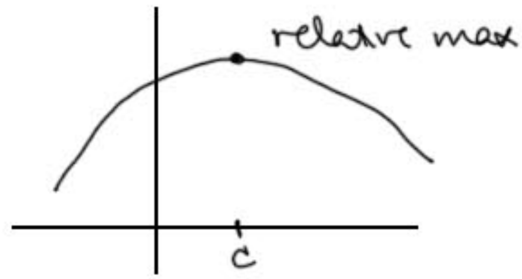
(use open intervals)

Concavity can also be used to help us find relative extrema:

## The Second Derivative Test for Relative Extrema

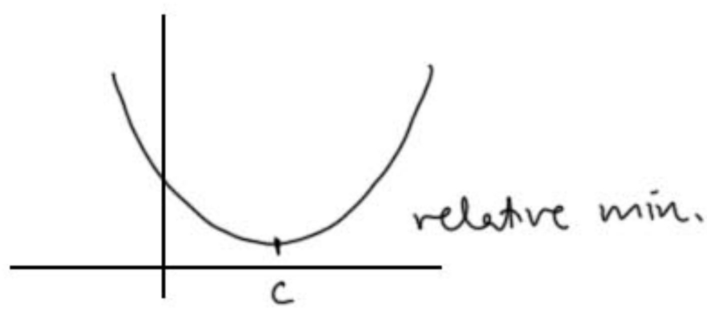
Suppose $f''$ is continuous near $x=c$.

if $\underbrace{f'(c)=0}_{\text{slope}=0}$ and $\underbrace{f''(c)<0}_{f \text{ is concave down at } x=c}$ then there is a relative max at $x = c$




if $\underbrace{f'(c) = 0}_{\text{slope} = 0}$ and $\underbrace{f''(c) > 0}_{\text{concave up.}}$ then

there is a relative minimum at $x = c$




Ex. from Lesson 18 + above $f(x) = x^4 - 4x^2 - 1$

Use The Second Derivative Test to find relative extrema.

So we find $x$- values where $f'(x) = 0$, and find The sign of $f''$ at These $x$-values.

* Note: The Second Derivative Test is only useful when all critical numbers have $f'(c) = 0$, and $f''(c) \neq 0$.

We have that here, since $f'(x) = 4x^3 - 8x$

$$4x(x^2-2) = 0$$

critical numbers: $x=0$, $x = \pm\sqrt{2}$

we know $f'(0) = 0$, $f'(\sqrt{2}) = 0$, $f'(-\sqrt{2}) = 0$

find $f''(x)$

$$f''(x) = 12x^2 - 8$$

for $x = 0$: $f'(0) = 0$ $\quad f''(0) = -8 < 0$

$\Rightarrow$ max at $x = 0$

for $x = \sqrt{2}$ $\quad f'(\sqrt{2}) = 0$ $\quad f''(\sqrt{2}) = 12(2) - 8 > 0$

$\Rightarrow$ min at $x = \sqrt{2}$

for $x = -\sqrt{2}$ $\quad f'(-\sqrt{2}) = 0$ $\quad f''(-\sqrt{2}) = 12(2) - 8 > 0$

$\Rightarrow$ min at $x = -\sqrt{2}$

Same info we got from the First Derivative Test in lesson 18.

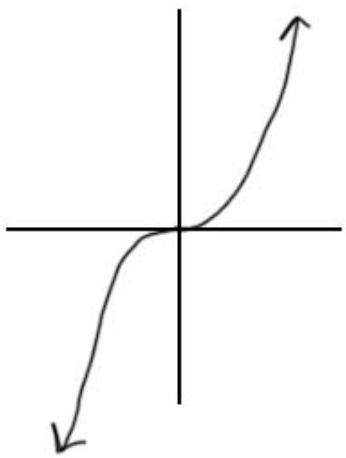
What happens with the Second Derivative Test when $f''(c) = 0$? No conclusion.

The examples below have $f''(c) = 0$ + no relative extrema, and $f''(c) = 0$ with a relative min.

Ex. $f(x) = x^3$

$f'(x) = 3x^2 = 0 \Rightarrow x = 0$ critical number.

$f''(x) = 6x \qquad f''(0) = 0$. can't use the Second Derivative Test.



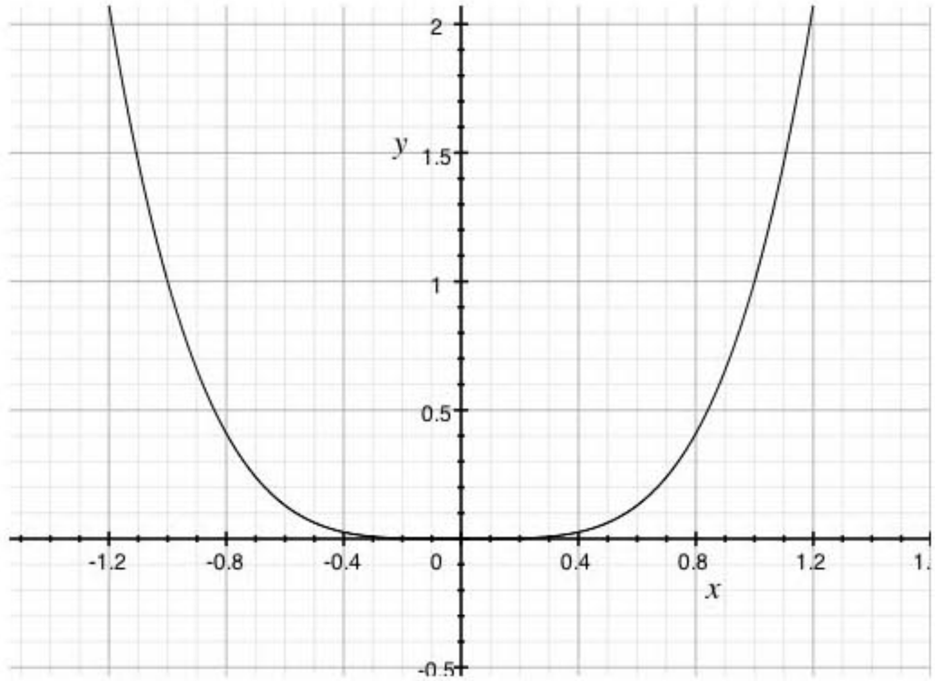
Zero slope at $x = 0$, but no relative extrema.

Ex. $f(x) = x^4$

$f'(x) = 4x^3 = 0 \Rightarrow x = 0$

$f''(x) = 12x^2 \qquad f''(0) = 0$ can't use the Second Derivative Test

note, we have a minimum at $x = 0$


2
1.5
1
0.5
-0.5
y
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
x