Math 20100
Calculus I

Lesson 18
Relative Extrema

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Relative Extrema

In lesson 16 we saw the following definitions for relative extrema:

Def. A function f has a relative maximum (or local maximum) at $x=c$ if

$f(c) \geq f(x)$ $\forall x$ in an open interval containing $x=c$.

Def. A function f has a relative minimum (or local minimum) at $x=c$ if

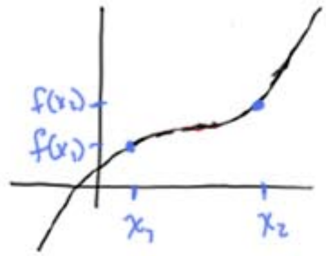
$f(c) \leq f(x)$ $\forall x$ in an open interval containing $x=c$.

In this lesson we learn how to find relative extrema.

<u>Definitions</u> f is <u>increasing</u> on interval (a,b)

if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ for $x_1, x_2 \in (a,b)$

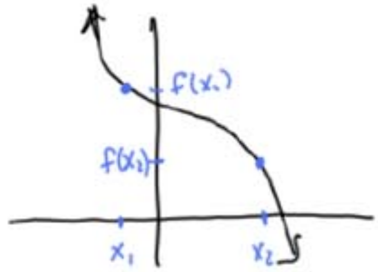
Ex.



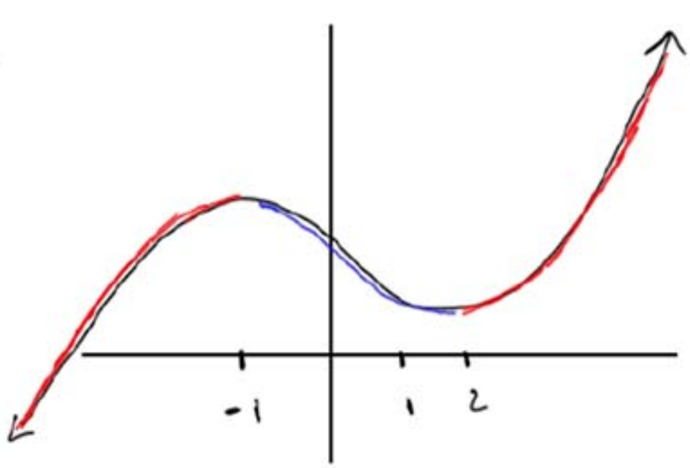
and f is <u>decreasing</u> on (a,b)

if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$ for $x_1, x_2 \in (a,b)$.

Ex.



Ex.



f is increasing on $(-\infty, -1) \cup (2, \infty)$

f is decreasing on $(-1, 2)$

Notice, If $f'(x) > 0$ on an interval,

f is increasing on that interval.

If $f'(x) < 0$ on an interval,

f is decreasing on that interval.

So the sign of f' will help us find intervals of increase and decrease of f, and help us find relative extrema.

Recall from Lesson 16:

<u>Def</u> $x = c$ is a <u>critical number</u> of f

$$\text{if: } \left.\begin{array}{l} f'(c) = 0 \\ \text{or } f'(c) \text{ DNE} \end{array}\right\} \text{ c must be in the domain of f.}$$

Call point $(c, f(c))$ a <u>critical point</u> of f.

To find the intervals on which f is increasing or decreasing, and to find relative extrema:

1) find any discontinuities of f (check the domain)

2) find $f'(x)$

3) find the critical numbers (x in the domain of f such that $f'(x)=0$ or $f'(x)$ DNE)

4) plot the critical numbers and discontinuities on a number line

5) find the sign of f' on each interval of the number line

6) if f' goes from negative to positive at the critical number $x=c$, f has a relative minimum at $x=c$.

if f' goes from positive to negative at the critical number $x=c$, f has a relative maximum at $x=c$.

step 6: "The First Derivative Test"

Ex. $f(x) = x^4 - 4x^2 - 1$ domain $\mathbb{R}$

$f'(x) = 4x^3 - 8x$ always exists

$4x^3 - 8x = 0$

$4x(x^2 - 2) = 0$

$x = 0 \quad | \quad x = \pm\sqrt{2}$

$f(\pm\sqrt{2}) = 4 - 4(2) - 1$
$= -5$

$f(0) = -1$

f: dec, min, inc, max, dec, min, inc

f': $-$, $+$, $-$, $+$

$-\sqrt{2}$ $\quad 0 \quad$ $\sqrt{2}$

$x = -2 \quad x = -1 \quad x = 1 \quad x = 2$

take one number in each interval + plug into f'

$f'(-2) = \underbrace{4(-2)}_{-}\underbrace{((-2)^2 - 2)}_{+} = -$

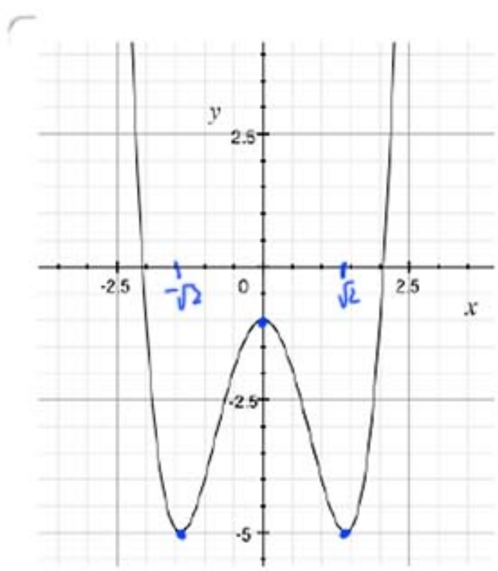
$f'(-1) = \underbrace{4(-1)}_{-}\underbrace{((-1)^2 - 2)}_{-} = +$

$f'(1) = \underbrace{4(1)}_{+}\underbrace{(1^2 - 2)}_{-} = -$

$f'(2) = \underbrace{4(2)}_{+}\underbrace{(2^2 - 2)}_{+} = +$

f has relative minimums at $x = \pm\sqrt{2}$ $\quad (\pm\sqrt{2}, -5)$

f has a relative maximum at $x = 0$ $\quad (0, -1)$



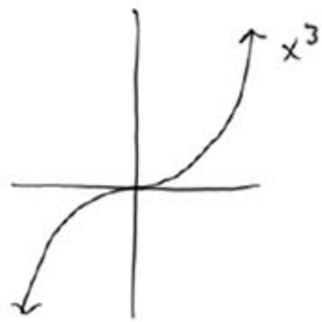
f is increasing on $(-\sqrt{2},0) \cup (\sqrt{2},\infty)$

f is decreasing on $(-\infty,-\sqrt{2}) \cup (0,\sqrt{2})$

Ex. $f(x) = x^3$

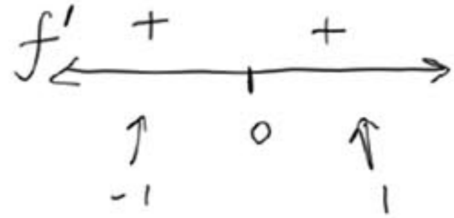
increasing

$(-\infty, \infty)$



$f'(x) = 3x^2 = 0$

$\Rightarrow x = 0$



no relative extrema

Ex. $f(x) = \dfrac{x^2}{x+1}$ Find the intervals on which f is increasing and decreasing, and find any relative extrema.

$f(-2) = \frac{(-2)^2}{-2+1} = \frac{4}{-1} = -4$

$f(0) = 0$

domain $x \neq -1$ $\quad \frac{(-1)^2}{0} \quad \frac{\text{nonzero}}{\text{zero}} \Rightarrow$ asymptote.

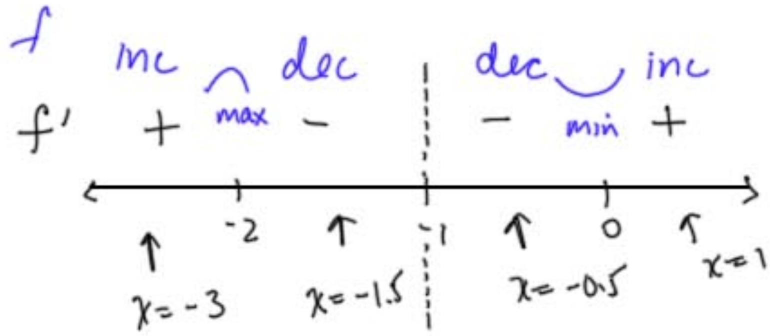
$$f'(x) = \frac{(x+1)(2x) - x^2(1)}{(x+1)^2} = \frac{2x^2+2x-x^2}{(x+1)^2}$$

$$= \frac{x^2+2x}{(x+1)^2} = 0 \Rightarrow x^2+2x=0$$

$$x(x+2)=0$$

$x=0,\ x=-2$ critical numbers.

$f'(-1)$ DNE but $x=-1$ asymptote of f not critical #



relative max at $x=-2$ $(-2,-4)$

relative min at $x=0$ $(0,0)$

$$f'(x) = \frac{x(x+2)}{(x+1)^2} \leftarrow \text{always} +$$

$$f'(-3) = \frac{-3(-3+2)}{+} = \frac{-(-)}{+} = +$$

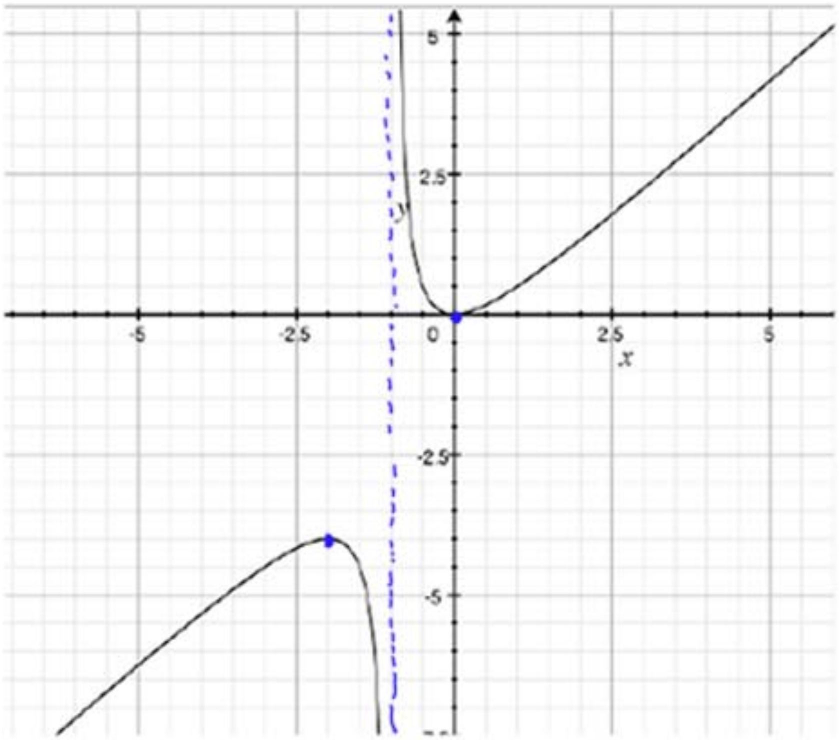
$$f'(-1.5) = \frac{-1.5(-1.5+2)}{+} = \frac{-+}{+} = -$$

$$f'(-0.5) = \frac{-0.5(-0.5+2)}{+} = \frac{-+}{+} = -$$

$$f'(1) = \frac{1(1+2)}{+} = \frac{+}{+} = +$$

f increasing on $(-\infty,-2) \cup (0,\infty)$

f decreasing on $(-2,-1) \cup (-1,0)$



Ex. $f(x) = x + \sqrt{1-x}$ Use The First Derivative Test to find all local extrema.

$f\left(\frac{3}{4}\right) = \frac{3}{4} + \sqrt{1-\frac{3}{4}}$

$= \frac{3}{4} + \sqrt{\frac{1}{4}}$

$\frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

domain of $f(x)$: need $1 - x \geq 0$

$+x \quad +x$

$1 \geq x \Rightarrow x \leq 1$

$f'(x) = 1 + \frac{1}{2}(1-x)^{-1/2}(-1)$

$= 1 - \frac{1}{2\sqrt{1-x}}$ DNE at $x = 1$

is in the domain of f, but not in domain of f'.

$\therefore$ $x=1$ is a critical number

Now, set $f'(x) = 0$:

$$1 - \frac{1}{2\sqrt{1-x}} = 0$$

$$\frac{1}{1} = \frac{1}{2\sqrt{1-x}}$$

$$2\sqrt{1-x} = 1$$

$$\sqrt{1-x} = \frac{1}{2}$$

$$1-x = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$1 - x = \frac{1}{4}$$

Critical #. $x = \frac{3}{4}$ check is in domain $x \leq 1$

be sure to show the restricted domain on the number line.

relative max at $x = 3/4$

$(3/4, 5/4)$

f inc max dec

f' $+$ $-$

$x=0$ $\frac{3}{4}$ $x=\frac{15}{16}$ 1

$$f'(x) = 1 - \frac{1}{2\sqrt{1-x}}$$

$$f'(0) = 1 - \frac{1}{2\sqrt{1}} = 1 - \frac{1}{2} = \frac{1}{2} = +$$

$$f'\left(\tfrac{15}{16}\right) = 1 - \frac{1}{2\sqrt{1-\frac{15}{16}}} = 1 - \frac{1}{2\sqrt{\frac{1}{16}}} = 1 - \frac{1}{2\cdot\frac{1}{4}} = 1 - 2 = -1$$

f increasing $(-\infty, 3/4)$ f decreasing $(\frac{3}{4}, 1)$.