Math 20100
Calculus I

Lesson 17
The Mean Value Theorem

Dr. A. Marchese, The City College of New York

Bookmarks have been added to this video
at the following times:

The Mean Value Theorem (MVT):

let f be a function with:

1) f continuous on the closed interval $[a,b]$

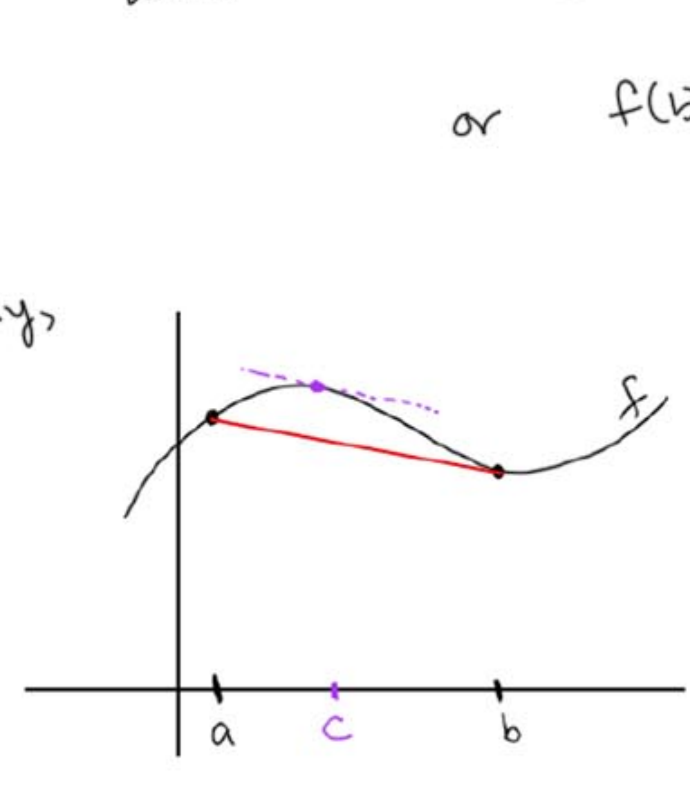
2) f differentiable on the open interval (a,b)

$$\text{then } \exists\, c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\exists$: there exists $\in$: in (an element of) $\ni$: such that

$$\text{or} \quad f(b)-f(a) = f'(c)(b-a).$$

Visually,

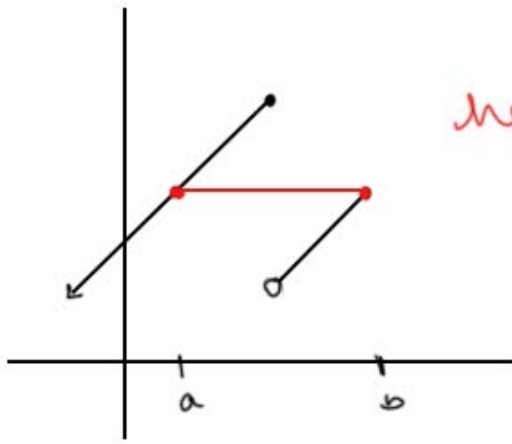


$$\text{secant slope} = \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

If conditions 1) + 2) are not satisfied, there may exist such a $c \in (a,b)$, but it is not guaranteed.

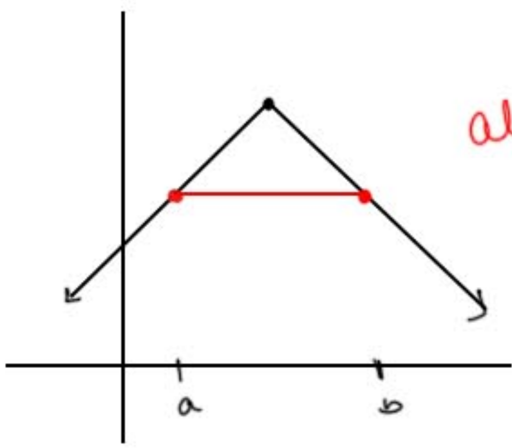
Ex. f <u>not</u> continuous on $[a,b]$



here $f(a) = f(b)$ so $\dfrac{f(b)-f(a)}{b-a} = 0$

but there are no points on f for which $f'(c) = 0$.

Ex. f <u>not</u> differentiable on (a,b)



also here, $f(a) = f(b)$ so $\dfrac{f(b)-f(a)}{b-a} = 0$

but there are no points on f for which $f'(c) = 0$.

Ex. Verify that the function satisfies the conditions of the Mean Value Theorem, and find all values of c guaranteed by the Theorem.

$f(x) = x^3 + x - 1$ on $[0,2]$

f is continuous on $\mathbb{R}$, so continuous on $[0,2]$.

$f'(x) = 3x^2+1$ exists on $\mathbb{R}$, so exists on $(0,2)$

$\therefore f$ is differentiable on $(0,2)$

$\therefore$ by MVT, $\exists\, c \in (0,2) \ni f'(c) = \dfrac{f(2)-f(0)}{2-0}$

$$3c^2+1 = \frac{9+1}{2-0} = \frac{10}{2} = 5$$

$$3c^2+1 = 5$$

$$3c^2 = 4 \quad c^2 = \frac{4}{3} \quad c = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$$

only $c = \frac{2}{\sqrt{3}} \in (0,2)$. $\boxed{c = \frac{2}{\sqrt{3}}}$.

$$\left(\text{we know } \frac{2}{\sqrt{3}} < \frac{2\sqrt{3}}{\sqrt{3}} = 2 \quad \text{since } \sqrt{3} > 1\right)$$

Ex. Verify that the function satisfies the conditions of the Mean Value Theorem, and find all values of c guaranteed by the Theorem.

$$f(x) = \frac{x}{x+2} \text{ on } [1,4]$$

Since $f(x)$ is a rational function, it is continuous on its domain, $(-\infty,-2)\cup(-2,\infty)$.

$[1,4]$ is contained in this domain.

$$f'(x) = \frac{(x+2)(1)-(x)(1)}{(x+2)^2} = \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2}$$ exists on the same domain,

so f is differentiable on $(1,4)$.

$\therefore$ by the MVT $\exists\, c \in (1,4) \ni f'(c) = \dfrac{f(4)-f(1)}{4-1}$

$$\frac{2}{(c+2)^2} = \frac{\frac{4}{6}-\frac{1}{3}}{4-1} = \frac{\frac{1}{3}}{3} = \frac{1}{9}$$

$$\frac{2}{(c+2)^2} = \frac{1}{9}$$ cross multiply

$$18 = (c+2)^2$$

$c+2 = \pm\sqrt{18}$ $\quad c = -2 \pm \sqrt{18}$ $\quad -2-\sqrt{18} < 0$
$\notin (1,4)$

notice $c = -2+\sqrt{18} \in (1,4)$ since

$$\sqrt{16} < \sqrt{18} < \sqrt{25}$$

$$4 < \sqrt{18} < 5$$

$$-2+4 < -2+\sqrt{18} < -2+5$$

$$2 < -2+\sqrt{18} < 3.$$

$$\therefore\ c = -2+\sqrt{18}$$

$$\boxed{c = -2+3\sqrt{2}}.$$

Ex. Show that the equation has exactly one real root:

$$2x-1-\sin x = 0.$$

Suppose a and b are roots of the equation.

let $f(x) = 2x-1-\sin x$

then $f(b) = f(a) = 0$.

we know $f(x)$ is continuous & differentiable on $\mathbb{R}$, so on $[a,b]$.

then by the MVT $\exists\, c \in (a,b) \ni$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{0}{b-a} = f'(c)$$

$$0 = f'(c)$$

$$f'(x) = 2-\cos x$$

$$f'(c) = 2-\cos(c) = 0$$

$\cos(c) = 2$ not possible

$\therefore$ there can not be 2 roots.

How do we know there are any roots?

Let's use the Intermediate Value Theorem:

notice $f(0) = -1$

$$f(\pi) = 2\pi - 1 - \sin\pi = 2\pi - 1 > 0$$

$\therefore \exists\, k \in (0,\pi) \ni f(k) = 0.$

$\therefore$ There is exactly one root of $2x - 1 - \sin x = 0$.