# Math 20100
# Calculus I

# Lesson 15
# Linear Approximations and Differentials

Dr. A. Marchese, The City College of New York

Bookmarks have been added to this video at the following times:

# Linear Approximations and Differentials

Let's start by looking at $f(x) = x^3 + 2x^2 - 1$, and finding an equation of the tangent line at $x = -1$.

point of tangency: $(-1, f(-1)) = (-1, 0)$

$$f(-1) = (-1)^3 + 2(-1)^2 - 1 = -1 + 2 - 1 = 0$$

$m = f'(-1)$

$f'(x) = 3x^2 + 4x \qquad f'(-1) = 3(-1)^2 + 4(-1)$

$$= 3(1) - 4 = -1 \qquad \therefore m = -1$$

$y - y_1 = m(x - x_1)$

$y - 0 = -1(x - (-1))$

$y = -x - 1$ equation of the tangent line.


We notice that if we zoom in close enough to the graph near $x=-1$, the tangent line is very close to the original function.

So for x values near $x=-1$, we can use the tangent line as an approximation to the original function (easier computation).

$f(x) = x^3+2x^2-1$ original

$y=-x-1$ tangent line at $x=-1$

For x-values near $x=-1$, $f(x) \approx -x-1$

approximately equal to

So if we want $f(-1.02)$, for example,

near $x=-1$

$f(-1.02) \approx \underbrace{-(-1.02)-1}_{\text{using tangent line instead of original } f(x)} = 1.02-1 = 0.02.$

Using The original function we can see

$f(-1.02) = 0.019592$ , so our approximate value is reasonable.

We say $f(x) \approx -x-1$ is The <u>linear approximation</u> to $f(x)$ near (at) $x=-1$.

also called The <u>linearization</u> of $f(x)$ at $x=-1$.

* Note, this is only useful for x-values near $x=-1$, the point of tangency.

notation:

$$y - y_1 = m(x - x_1)$$ tangent line to $f(x)$ at $x=a$

$$m = f'(a) \qquad (x_1, y_1) = (a, f(a))$$

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$ tangent line

$f(x) \approx f(a) + f'(a)(x-a)$ linearization of $f$ at $x=a$.

Ex. Use a linear approximation to estimate $\sqrt{99.8}$.

$f(x) = ?$ point $x=a$ ?

Let $f(x) = \sqrt{x}$ and let $a=100$ since $f(a) = \sqrt{100} = 10$, easy computation.

then the point of tangency is $(100, 10)$
$(a, f(a))$

and $m = f'(100)$.

$f'(x) = \frac{1}{2\sqrt{x}}$ $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05$

then $f(x) \approx f(a) + f'(a)(x-a)$

$$f(x) \approx 10 + 0.05(x-100)$$

and $\sqrt{99.8} = f(99.8) \approx 10 + 0.05(99.8-100)$

$= 10 + 0.05(-0.2)$

$$= 10 - 0.010 = 9.99$$

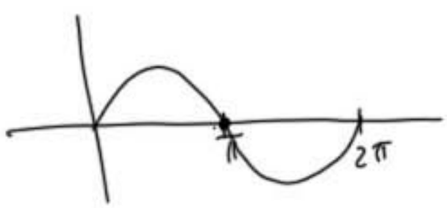
$$\therefore \sqrt{99.8} \approx 9.99.$$

Ex. Use a linear approximation to estimate

$\sin(3)$. ← 3 radians.

$f(x) = ?$ point $x = a$?

$f(x) = \sin x$ point of tangency $a = \pi$ $\pi \approx 3.14$ near 3




point $(\pi, \sin(\pi))$

$= (\pi, 0)$

$(a, f(a))$

$f'(x) = \cos x$ $f'(\pi) = \cos \pi = -1$ = slope of tangent line

$$f(x) \approx 0 + -1(x - \pi)$$

so $f(x) \approx -x + \pi$ near $x = \pi$

$\sin x \approx -x + \pi$ near $x = \pi$

$\sin 3 \approx -3 + \pi$.

With some new notation, we can use the idea of the linear approximation to get a quick estimate of how errors propogate in computations.

## Differentials $dx$ and $dy$

So far we've seen $\frac{dy}{dx} = \frac{d}{dx}(y) = f'(x)$, but

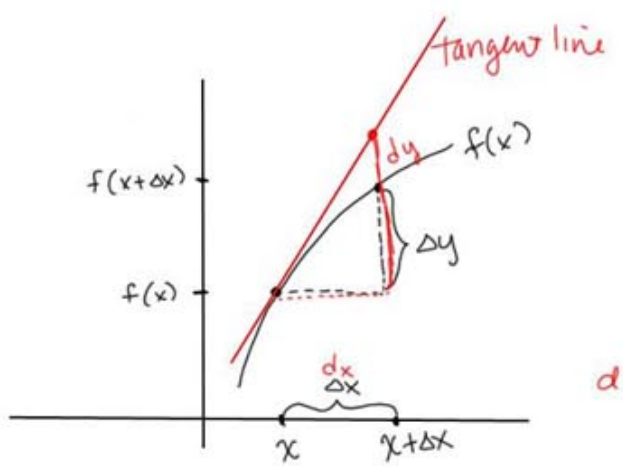
$dy$ and $dx$ have no meaning on their own.

Now we introduce $dx$ as an independent variable (it can take any value) and then for $y = f(x)$

we define $dy = f'(x)\,dx$

Then ① $dy \div dx = f'(x) = \frac{dy}{dx}$ so the notation is consistent

②




approximating f(x) by the tangent line

$dx = \Delta x$

$dy$ is the change in y-value when using the tangent line approximation.

This will be an approximation to $\Delta y$, the actual change in function value.

Note, the linear approximation uses the differential: $f(x) \approx f(a) + \underbrace{f'(a)\underbrace{(x-a)}_{\Delta x}}_{\text{differential}}$

We use differentials when we're more interested in $\Delta y$, the change in y-values (function values) for a given $\Delta x$ (change in x-values). We use the differential $dy$ to approximate $\Delta y$.

Ex. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.

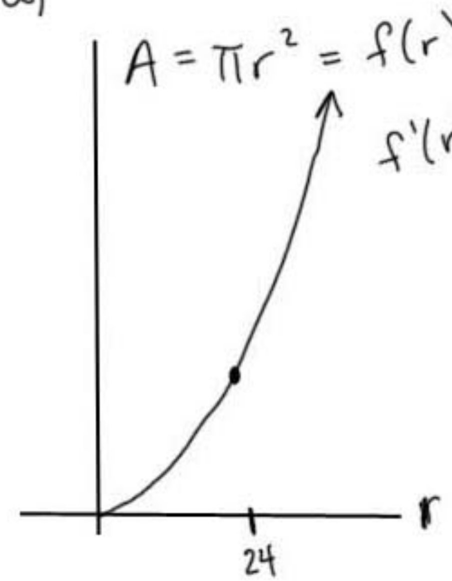
a) Use differentials to estimate the maximum error in the calculated area of the disk.

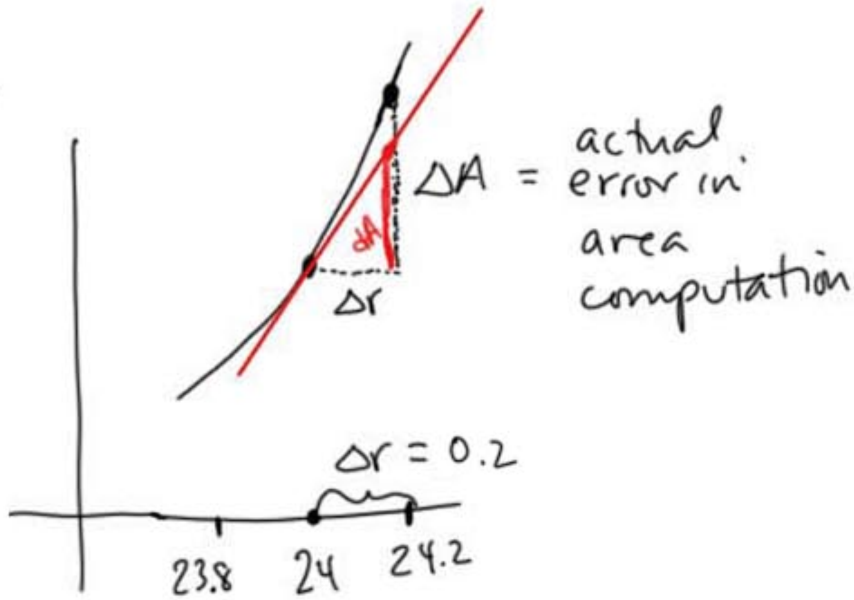
b) What is the relative error? What is the percentage error?

given: $r = 24$ cm $\quad \underbrace{\text{max error}}$ of $0.2$ cm

$$\Delta r = 0.2 \text{ cm}$$

a)




$$dr = \Delta r$$

$$\Delta A \approx dA = f'(r)\,dr = f'(r)\,\Delta r$$

$$= 2\pi r \,\Delta r$$

$$= 2\pi(24)(0.2)$$

$$= 9.6\pi \text{ cm}^2.$$

approximation for maximum error in area

$$\left( \text{actual } \Delta A = \underbrace{\pi(24.2)^2}_{f(24.2)} - \underbrace{\pi(24)^2}_{f(24)} = 9.64\pi \text{ cm}^2 \right)$$

b) <u>relative error</u> in area $= \dfrac{\Delta A}{A}$ ← error in area / ← area

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{9.6\pi}{\pi(24)^2} \approx 0.017$$

percentage error : change relative area into %

$$= 1.7\%.$$