Math 20100
Calculus I

Lesson 14
Related Rates

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Related Rates

When quantities are related (for example, the radius and volume of a sphere), their rates of change are also related.

Ex. A spherical balloon is deflating at a rate of 20 cm³/s. At what rate is the radius of the balloon decreasing when the radius is 5 cm?

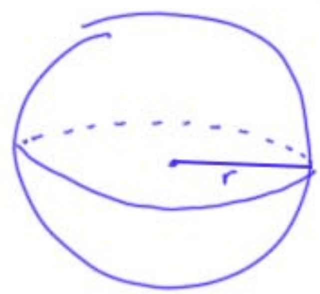
cm^3 = volume

$\frac{dV}{dt} = -20$

find $\left.\frac{dr}{dt}\right|_{r=5}$

V = volume of sphere

Step 1: sketch



step 2: assign a variable to any quantity that is changing with time

step 3: find an equation relating the variables from step 2.

$$V = \frac{4}{3}\pi r^3$$

step 4: take the derivative (with respect to t) of the equation in step 3.

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} \qquad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

step 5: see which derivatives (rates) are given in the problem. plug them in.

given $\frac{dV}{dt} = -20$

$$-20 = 4\pi r^2 \frac{dr}{dt} \qquad -5 = \pi r^2 \frac{dr}{dt}$$

step 6: Solve the problem by plugging in any given variable values, and solving for the rate of change.

given $r=5$

find $\left.\frac{dr}{dt}\right|_{r=5}$

$$-5 = \pi (5)^2 \left.\frac{dr}{dt}\right|_{r=5}$$

$$-5 = 25\pi \left.\frac{dr}{dt}\right|_{r=5}$$

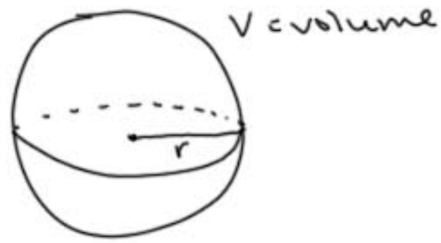
radius decreasing

$$\therefore \left.\frac{dr}{dt}\right|_{r=5} = \frac{-1}{5\pi} \ \frac{cm}{s}$$

step 7: make sure to answer the question.

The radius is decreasing at a rate of $\frac{1}{5\pi} \frac{cm}{s}$.

Ex. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?



$$V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot \frac{d}{dt}(r^3)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

given $\frac{dr}{dt} = 4 \frac{mm}{s}$

$$\frac{dV}{dt} = 4\pi r^2 (4) = 16\pi r^2$$

Asked for $\left.\frac{dV}{dt}\right|_{d=80mm}$

↑ means $r = 40mm$

$$\left.\frac{dV}{dt}\right|_{r=40} = 16\pi (40)^2 =$$

$$25{,}600\pi \ \frac{mm^3}{s}.$$

When the diameter is 80 mm, the volume is increasing at a rate of $25{,}600\pi \ mm^3/s$.

Ex. The length of a rectangle is increasing at a rate of 2 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm,

a) how fast is the perimeter increasing?

b) how fast is the area increasing?

P = perimeter

A = area

a) $P = 2l + 2w$

$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

given $\frac{dl}{dt} = 2\text{cm/s}$, $\frac{dw}{dt} = 3\text{cm/s}$

$$\frac{dP}{dt} = 2(2) + 2(3) = 10\ \frac{\text{cm}}{\text{s}}$$

note: does not matter what l & w are, $\frac{dP}{dt}$ is constant, always 10 cm/s.

$\therefore$ When the length = 20cm and the width = 10cm, the perimeter increases at a rate of 10cm/s.

b) $A = l \cdot w \Rightarrow \frac{dA}{dt} = \frac{dl}{dt} \cdot w + l \cdot \frac{dw}{dt}$ (product rule)

given $\frac{dl}{dt} = 2\,cm/s$, $\frac{dw}{dt} = 3\,cm/s \Rightarrow \frac{dA}{dt} = 2w + 3l$

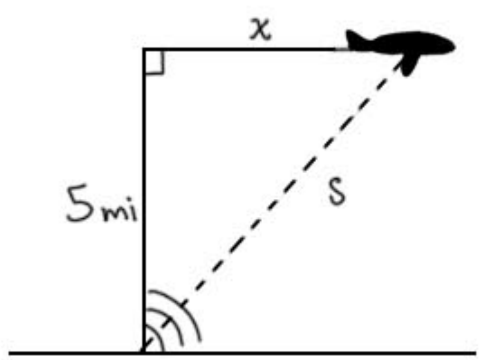
note: $\frac{dA}{dt}$ depends on w & l, not constant.

We are asked to find $\left.\frac{dA}{dt}\right|_{\substack{l=20 \\ w=10}} = 2(10) + 3(20)$

$$= 80 \ \frac{cm^2}{s}$$

$\therefore$ When the length = 20cm and the width = 10cm, the area increases at a rate of 80 cm^2/s.

Ex. An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna. When the plane is 10 miles from the antenna, $s = 10$ the radar detects that the distance between the antenna and plane is increasing at a rate of 240 miles per hour. $\left.\frac{ds}{dt}\right|_{s=10} = 240\,\frac{mi}{h}$ What is the speed of the plane at that time? $\left.\frac{dx}{dt}\right|_{s=10}$



$$x^2 + 5^2 = s^2$$
$$x^2 + 25 = s^2$$
$$\frac{d}{dt}\left(x^2 + 25\right) = \frac{d}{dt}\left(s^2\right)$$

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

given $\left.\frac{ds}{dt}\right|_{s=10} = 240 \frac{mi}{h}$

$$2x\left.\frac{dx}{dt}\right|_{s=10} = 2(10)(240)$$

we need x when $s=10$.

$x^2+25=s^2$

$x^2+25=10^2$

$x^2=75$

$x=\pm\sqrt{75}=\pm 5\sqrt{3}$

we have $x>0 \Rightarrow x=5\sqrt{3}$

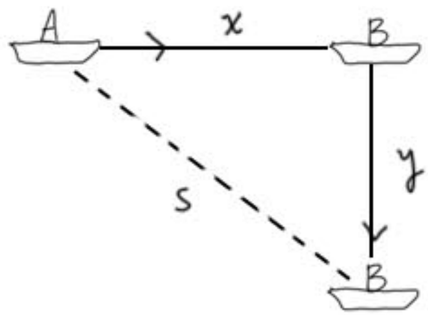
$$2(5\sqrt{3})\left.\frac{dx}{dt}\right|_{s=10} = 4800$$

$$\left.\frac{dx}{dt}\right|_{s=10} = \frac{4800}{10\sqrt{3}} = \frac{480}{\sqrt{3}} = \frac{480\sqrt{3}}{3}$$

$$= 160\sqrt{3}\ \frac{mi}{h} \approx 277\ \frac{mi}{h}$$

$\therefore$ The speed of the plane is $160\sqrt{3}\ \frac{mi}{h}$ at that time.

Ex. At noon, ship A is 100 km west of ship B. Ship A is traveling east at 35 km/h, and ship B is traveling south at 10 km/h. How fast is the distance between the ships changing at 2pm?



$$x^2 + y^2 = s^2$$

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(s^2)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$\div 2$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = s\frac{ds}{dt}$$

given $\frac{dx}{dt} = -35$ km/h

& $\frac{dy}{dt} = 10$ km/h $\Rightarrow$ $-35x + 10y = s\frac{ds}{dt}$.

We are asked for $\left.\frac{ds}{dt}\right|_{2pm}$. So we need the x, y, and s values at 2pm.

If ship A moves east at 35 km/h, after 2 hours it moved 70 km. $\therefore x = 100 - 70 = 30$ km.

If ship B moves south at 10 km/h, after 2 hours it moved 20 km. $\therefore y = 0 + 20 = 20$ km.

& $x^2 + y^2 = s^2$ so $30^2 + 20^2 = s^2$

$s^2 = 900 - 400 = 500$ $\quad s = \sqrt{500} = 10\sqrt{5}$.
≈ 22 km

So $\quad -35x + 10y = s\frac{ds}{dt}$

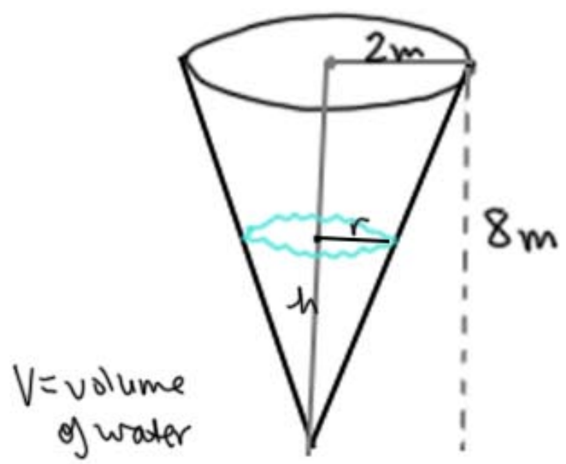
$$-35(30) + 10(20) = 10\sqrt{5}\left.\frac{ds}{dt}\right|_{2pm}$$

$$-1050 + 200 = 10\sqrt{5}\left.\frac{ds}{dt}\right|_{2pm}$$

$$\left.\frac{ds}{dt}\right|_{2pm} = \frac{-850}{10\sqrt{5}} = -\frac{85\sqrt{5}}{5} = -17\sqrt{5} \approx -38 \frac{km}{h}.$$

∴ The distance between the ships is decreasing at a rate of $17\sqrt{5}$ km/h at 2pm.

Ex. Water is leaking out of an inverted conical tank causing the height of the water to decrease by 20 cm/min. The top of the tank has a diameter of 4m and the height is 8m. Find the rate at which the water is leaking (ie $\frac{dV}{dt}$) when the height of the water is 1 m.



r = radius at top of water

h = height of water

$$V = \frac{1}{3}\pi r^2 h$$

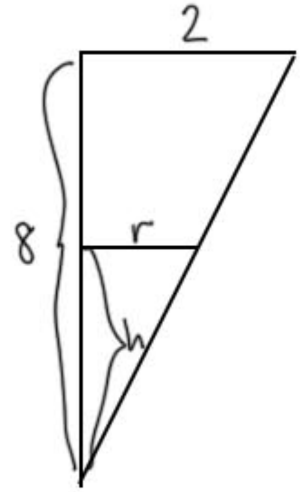
notice we are given $\frac{dh}{dt} = 20 \frac{cm}{min} = .02 \frac{m}{min}$

and we are asked for $\left.\frac{dV}{dt}\right|_{h=1}$. Since we're not given

any info about $\frac{dr}{dt}$, and we are given dimensions

for the tank, we replace r by what it equals in

terms of h:



$$\frac{2}{8} = \frac{r}{h} \Rightarrow r = \frac{2h}{8} = \frac{1}{4}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$$

$$V = \frac{\pi}{48}h^3$$

Now, continue...

STOP Work on this problem on your own

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{48} h^3\right)$$

$$\frac{dV}{dt} = \frac{\pi}{48} \cdot 3h^2 \frac{dh}{dt}$$

$$\therefore \quad \frac{dV}{dt} = \frac{\pi}{16} h^2 \frac{dh}{dt}$$

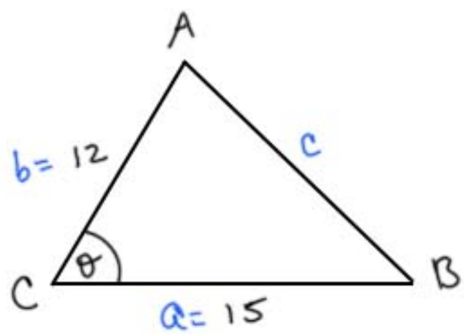
$$\frac{dV}{dt} = \frac{\pi}{16} h^2 (.02)$$

$$\left.\frac{dV}{dt}\right|_{h=1} = \frac{\pi}{16}(1)^2(.02) = \frac{.02}{16}\pi = \frac{2}{1600}\pi = \frac{\pi}{800}\ \frac{m^3}{min}$$

$$= \frac{\pi}{800}\ 100^3\ \frac{cm^3}{min} = 1250\pi\ \frac{cm^3}{min} \approx 3927\ \frac{cm^3}{min}.$$

$\therefore$ when the height of the water is 1 m, the water is leaking at a rate of $1250\pi\ \frac{cm^3}{min}$.

Ex. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of 0.1 rad/s. How fast is the length of the third side increasing when the angle between the sides of fixed length is $\frac{\pi}{3}$?



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta$$

$$c^2 = 369 - 360\cos\theta$$

$$\frac{d}{dt}(c^2) = \frac{d}{dt}(369 - 360\cos\theta)$$

$$2c\frac{dc}{dt} = -360(-\sin\theta)\frac{d\theta}{dt}$$

$$2c\frac{dc}{dt} = 360\sin\theta\frac{d\theta}{dt}$$

we are given $\frac{d\theta}{dt} = 0.1\,\text{rad}/s$

$$2c\frac{dc}{dt} = 360\sin\theta\,(0.1)$$

$$2c\frac{dc}{dt} = 36\sin\theta$$

$$c\frac{dc}{dt} = 18\sin\theta$$

asked to find $\left.\frac{dc}{dt}\right|_{\theta = \frac{\pi}{3}}$. notice we'll need the c-value when $\theta = \frac{\pi}{3}$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 12^2 + 15^2 - 2(12)(15)\cos\frac{\pi}{3}$$

$$= 369 - 360\left(\tfrac{1}{2}\right) = 369 - 180 = 189 \quad \Rightarrow c = \sqrt{189} = 3\sqrt{21}$$

So $\quad c\dfrac{dc}{dt} = 18\sin\theta$

$$3\sqrt{21}\left.\frac{dc}{dt}\right|_{\theta=\frac{\pi}{3}} = 18\sin\frac{\pi}{3}$$

$$\left.\frac{dc}{dt}\right|_{\theta=\frac{\pi}{3}} = 18\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{3\sqrt{21}} = \frac{3\sqrt{3}}{\sqrt{21}} = \frac{3\sqrt{3}\sqrt{21}}{21} = \frac{3\sqrt{7}}{7}\ \frac{m}{s}.$$

$\therefore$ When the angle between the fixed sides is $\frac{\pi}{3}$, the third side is increasing at a rate of $\frac{3\sqrt{7}}{7}\ \frac{m}{s}$.