Math 20100
Calculus I

Lesson 13
Implicit Differentiation

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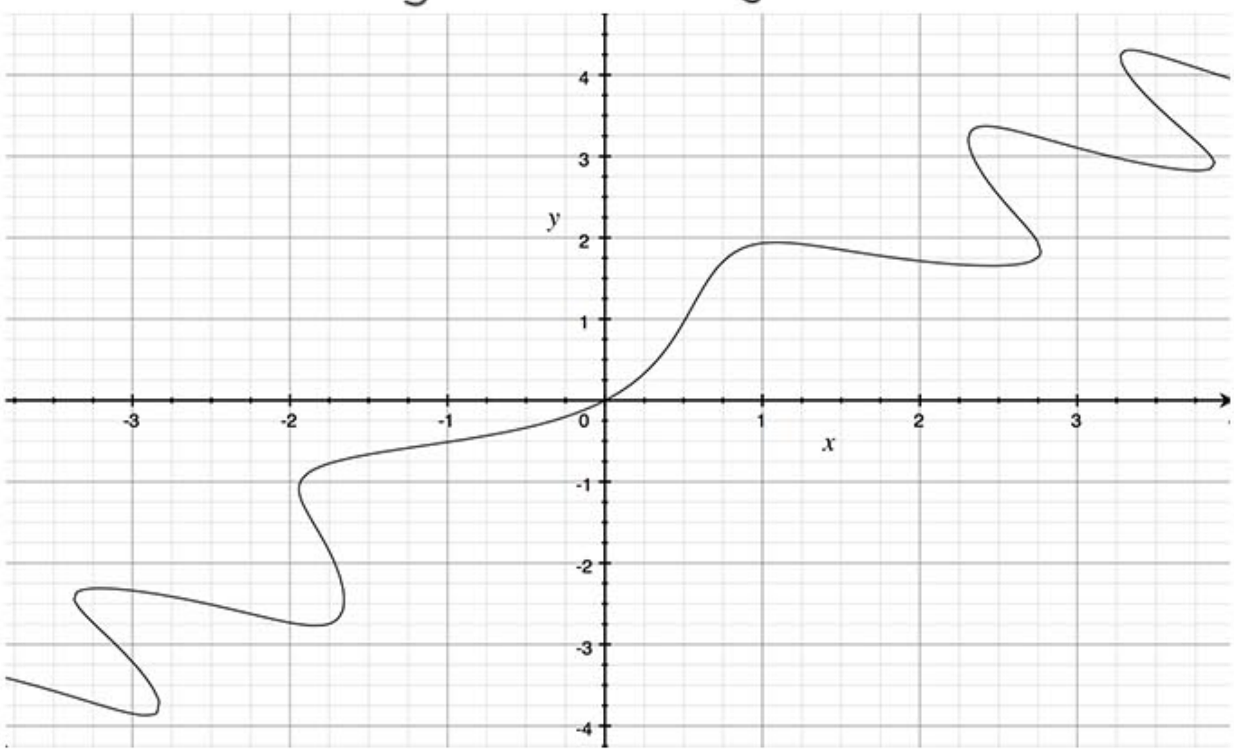
Implicit Differentiation

Until now we've only considered when $y = f(x)$, ie the relationship between x + y is such that y is a function of x and the equation can be solved for y.

But consider $y - x = \sin(xy)$



To find its slope at any point, we use implicit differentiation.

We take the derivative with respect to x of both sides of the equation:

$$y - x = \sin(xy)$$

$$\frac{d}{dx}(y - x) = \frac{d}{dx}\left(\sin(xy)\right)$$

$$\frac{d}{dx}(y) - \frac{d}{dx}(x) = \cos(xy) \cdot \underbrace{\frac{d}{dx}(xy)}_{\text{product rule}}$$

$$\frac{dy}{dx} - 1 = \cos(xy) \cdot \left(\underbrace{\frac{d}{dx}(x)}_{1} \cdot y + x \cdot \frac{d}{dx}(y)\right)$$

$$\frac{dy}{dx} - 1 = \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

need to solve for $\frac{dy}{dx}$, so distribute & simplify

$$\frac{dy}{dx} - 1 = y\cos(xy) + x\frac{dy}{dx}\cos(xy)$$

$$+1 \qquad\qquad +1$$

$$-x\frac{dy}{dx}\cos(xy) \qquad\qquad -x\frac{dy}{dx}\cos(xy)$$

$$\frac{dy}{dx} - x\frac{dy}{dx}\cos(xy) = y\cos(xy) + 1$$

$$\frac{dy}{dx}\left(1 - x\cos(xy)\right) = y\cos(xy) + 1$$

$$\frac{dy}{dx} = \frac{y\cos(xy)+1}{1-x\cos(xy)}.$$

So, for example, we know this curve goes through the origin since $0-0=\sin(0)$

And the slope at $(0,0)$ is

$$\left.\frac{dy}{dx}\right|_{(0,0)} = \frac{0\cdot\cos(0)+1}{1-0\cdot\cos(0)} = 1.$$

Ex. Find $\frac{dy}{dx}$ for $y^5 + x^2y^3 = 1 + x^4y$

$$\frac{d}{dx}\left(y^5+x^2y^3\right) = \frac{d}{dx}\left(1+x^4y\right)$$

$$\frac{d}{dx}(y^5) + \frac{d}{dx}(x^2y^3) = \frac{d}{dx}(1) + \frac{d}{dx}(x^4y)$$

$$5y^4\frac{dy}{dx} + \frac{d}{dx}(x^2)\cdot y^3 + x^2\cdot\frac{d}{dx}(y^3) = 0 + \frac{d}{dx}(x^4)\cdot y + x^4\cdot\frac{d}{dx}(y)$$

$$5y^4\frac{dy}{dx} + 2xy^3 + x^2\cdot 3y^2\frac{dy}{dx} = 4x^3y + x^4\frac{dy}{dx}$$

$$5y^4\frac{dy}{dx} + 3x^2y^2\frac{dy}{dx} - x^4\frac{dy}{dx} = 4x^3y - 2xy^3$$

$$\frac{dy}{dx}\left(5y^4 + 3x^2y^2 - x^4\right) = 4x^3y - 2xy^3$$

$$\frac{dy}{dx} = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}.$$

We know the point $(0,1)$ is on this curve,

and the slope at $(0,1)$ is

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{4(0) - 2(0)}{5(1) - 3(0) - 0} = \frac{0}{5} = 0.$$

Ex. Find the points on the curve where the tangent line is horizontal: $\frac{dy}{dx} = 0$

$$x^2 - xy + y^2 - 1 = 0$$

$$\frac{d}{dx}\left(x^2 - xy + y^2 - 1\right) = \frac{d}{dx}(0)$$

$$2x - \frac{d}{dx}(xy) + 2y\frac{dy}{dx} - 0 = 0$$

$$2x - \left(\frac{d}{dx}(x)\cdot y + x\cdot\frac{d}{dx}(y)\right) + 2y\frac{dy}{dx} = 0$$

$$2x - \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x - y = x\frac{dy}{dx} - 2y\frac{dy}{dx}$$

$$2x - y = \frac{dy}{dx}(x - 2y)$$

$$\frac{dy}{dx} = \frac{2x-y}{x-2y} \overset{\text{set}}{=} 0 \Rightarrow 2x - y = 0$$

$$y = 2x$$

plug into original:

$$x^2 - xy + y^2 - 1 = 0$$

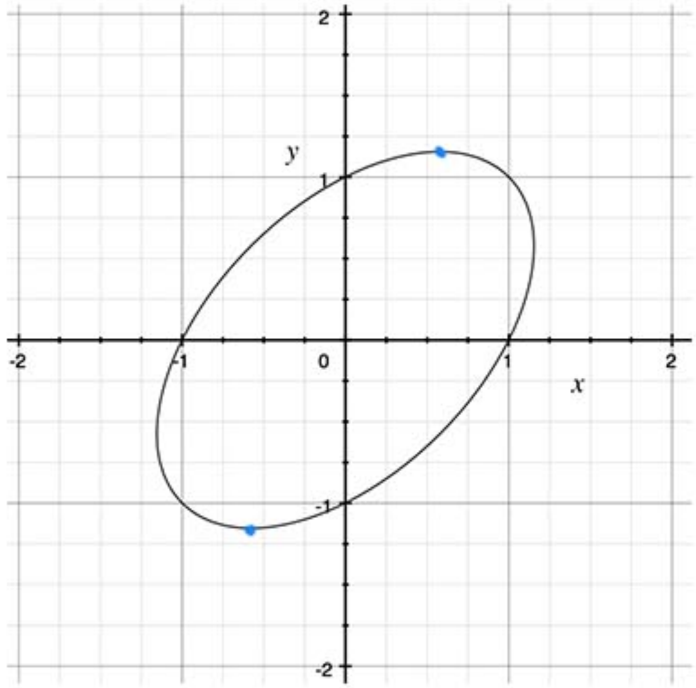
$$x^2 - x(2x) + 4x^2 - 1 = 0$$

$$3x^2 - 1 = 0$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3} \quad x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$$

$\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ + $\left(\frac{-1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$ since $y = 2x$.



Ex. Find $\frac{dy}{dx}$ for $\tan\left(\frac{x}{y}\right) = x+y$

$$\frac{d}{dx}\left(\tan\left(\frac{x}{y}\right)\right) = \frac{d}{dx}(x+y)$$

$$\sec^2\left(\frac{x}{y}\right) \cdot \frac{d}{dx}\left(\frac{x}{y}\right) = 1 + \frac{dy}{dx}$$

$$\sec^2\left(\frac{x}{y}\right) \cdot \left(\frac{y(1) - x \cdot \frac{dy}{dx}}{y^2}\right) = 1 + \frac{dy}{dx}$$

Mult by y^2 to get rid of the fraction easier to solve for $\frac{dy}{dx}$

$$\sec^2\left(\frac{x}{y}\right)\left(y - x\frac{dy}{dx}\right) = y^2 + y^2\frac{dy}{dx}$$

$$y\sec^2\left(\frac{x}{y}\right) - x\frac{dy}{dx}\sec^2\left(\frac{x}{y}\right) = y^2 + y^2\frac{dy}{dx}$$

$$y\sec^2\left(\frac{x}{y}\right) - y^2 = y^2\frac{dy}{dx} + x\frac{dy}{dx}\sec^2\left(\frac{x}{y}\right)$$

$$y\sec^2\left(\frac{x}{y}\right) - y^2 = \frac{dy}{dx}\left(y^2 + x\sec^2\left(\frac{x}{y}\right)\right)$$

$$\therefore \quad \frac{dy}{dx} = \frac{y\sec^2\left(\frac{x}{y}\right) - y^2}{y^2 + x\sec^2\left(\frac{x}{y}\right)}.$$