Math 20100
Calculus I

Lesson 12
The Chain Rule

Dr. A. Marchese, The City College of New York

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The Chain Rule

In this lesson we learn how to differentiate a composition of functions $f(g(x))$.

The Chain Rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$.

If we let $y = f(u)$ and $u = g(x)$, we can write the chain rule as $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Ex. $y = (x^2+3)^5$ here $y = f(u) = u^5$

$u = g(x) = x^2+3$

$f'(u) = 5u^4$

$g'(x) = 2x$

so $y' = f'(g(x)) \cdot g'(x)$

$= 5(x^2+3)^4 \cdot 2x$

$= 10x(x^2+3)^4$.

Note: In general, for $y = (g(x))^n$

$$y' = n(g(x))^{n-1} \cdot g'(x).$$

(combines the power rule & chain rule.)

Ex. $y = \sqrt{x^2+2x} = (x^2+2x)^{1/2}$

$$y' = \tfrac{1}{2}(x^2+2x)^{-1/2} \cdot (2x+2)$$

$$= \frac{x+1}{\sqrt{x^2+2x}}.$$

Ex. $y = \sqrt[3]{\sin x + 2} = (\sin x + 2)^{1/3}$

$$y' = \tfrac{1}{3}(\sin x + 2)^{-2/3}(\cos x)$$

$$= \frac{\cos x}{3(\sin x + 2)^{2/3}}.$$

Ex. $y = (\sin x + \cos x)^6$

STOP Work on this problem on your own

$$y' = 6(\sin x + \cos x)^5 (\cos x - \sin x).$$

Ex. $y = x^2 + \sin^2 x$

$$= x^2 + (\sin x)^2$$

$$y' = 2x + 2(\sin x)(\cos x)$$

Ex. $y = \cos(\pi x)$

$$y' = -\sin(\pi x) \cdot \frac{d}{dx}(\pi x)$$

$$= -\sin(\pi x) \cdot \pi$$

$$= -\pi \sin(\pi x).$$

Ex. $y = \dfrac{2t}{\sqrt{t^2-9}} = \dfrac{2t}{(t^2-9)^{1/2}}$ composition in a quotient

$\Rightarrow$ we'll need the chain rule within the quotient rule.

$$y' = \frac{(t^2-9)^{1/2}\cdot\frac{d}{dt}(2t) - 2t\cdot\overbrace{\frac{d}{dt}\left((t^2-9)^{1/2}\right)}^{\text{chain rule}}}{\left((t^2-9)^{1/2}\right)^2}$$

$$= \frac{(t^2-9)^{1/2}(2) - \cancel{2}t\cdot\overbrace{\cancel{\tfrac{1}{2}}(t^2-9)^{-1/2}(2t)}}{t^2-9}\cdot\frac{(t^2-9)^{1/2}}{(t^2-9)^{1/2}}$$

$$= \frac{2(t^2-9) - 2t^2}{(t^2-9)^{3/2}} = \frac{-18}{(t^2-9)^{3/2}}$$

Ex. $y = (t+1)^{2/3}(2t^2-1)^3$

$$y' = \overbrace{\frac{d}{dt}\left((t+1)^{2/3}\right)}^{\text{chain rule}}\cdot(2t^2-1)^3 + (t+1)^{2/3}\cdot\overbrace{\frac{d}{dt}\left((2t^2-1)^3\right)}^{\text{chain rule}}$$

$$= \overbrace{\tfrac{2}{3}(t+1)^{-1/3}(1)}\cdot(2t^2-1)^3 + (t+1)^{2/3}\overbrace{\left(3(2t^2-1)^2(4t)\right)}$$

$$= \frac{2(2t^2-1)^3}{3(t+1)^{1/3}} + 12t(t+1)^{2/3}(2t^2-1)^2.$$

Ex. $y = \cos^4(\sin^3(\pi x))$

$$= \left(\cos(\sin^3(\pi x))\right)^4$$

$$= \left(\cos\left((\sin(\pi x))^3\right)\right)^4$$

$$y' = 4\left(\cos\left((\sin\pi x)^3\right)\right)^3 \cdot \frac{d}{dx}\left(\cos\left((\sin\pi x)^3\right)\right)$$

$$= 4\left(\cos\left((\sin\pi x)^3\right)\right)^3 \cdot \left(-\sin\left((\sin\pi x)^3\right)\right) \cdot \frac{d}{dx}\left((\sin\pi x)^3\right)$$

$$= 4\left(\cos\left((\sin\pi x)^3\right)\right)^3 \cdot \left(-\sin\left((\sin\pi x)^3\right)\right) \cdot 3(\sin\pi x)^2 \cdot \frac{d}{dx}(\sin\pi x)$$

$$= 4\left(\cos\left((\sin\pi x)^3\right)\right)^3 \cdot \left(-\sin\left((\sin\pi x)^3\right)\right) \cdot 3(\sin\pi x)^2 \cdot \cos(\pi x) \cdot \underbrace{\frac{d}{dx}(\pi x)}_{\pi}$$

$$= -12\pi \cos^3(\sin^3\pi x) \cdot \sin(\sin^3\pi x) \cdot \sin^2\pi x \cdot \cos\pi x.$$

Ex. $y = \left[x^2 + (1-3x)^4 \right]^3$ find y'.

Work on this problem on your own

$$y' = 3\left[x^2 + (1-3x)^4 \right]^2 \cdot \frac{d}{dx}\left[x^2 + (1-3x)^4 \right]$$

$$= 3\left[x^2 + (1-3x)^4 \right]^2 \cdot \left[2x + 4(1-3x)^3 \cdot \underbrace{\frac{d}{dx}(1-3x)}_{-3} \right]$$

$$= 3\left[x^2 + (1-3x)^4 \right]^2 \cdot \left[2x - 12(1-3x)^3 \right].$$

Ex. $y = \left(\frac{2x+2}{x-1} \right)^3$ quotient inside chain

$$y' = 3\left(\frac{2x+2}{x-1} \right)^2 \cdot \frac{d}{dx}\left(\frac{2x+2}{x-1} \right)$$

$$= 3\left(\frac{2x+2}{x-1} \right)^2 \cdot \frac{(x-1)(2) - (2x+2)(1)}{(x-1)^2}$$

$$= 3\left(\frac{2x+2}{x-1} \right)^2 \left(\frac{2x-2-2x-2}{(x-1)^2} \right)$$

$$= 3\left(\frac{2x+2}{x-1}\right)^2\left(\frac{-4}{(x-1)^2}\right) = \frac{-12(2x+2)^2}{(x-1)^4}.$$

Ex. Where does $f(x) = \cos^2(\pi x)$ have a horizontal tangent line?

Asking where $f'(x) = 0$.

$$f(x) = \left(\cos(\pi x)\right)^2$$

$$f'(x) = 2\left(\cos(\pi x)\right)\cdot\frac{d}{dx}\left(\cos(\pi x)\right)$$

$$= 2\left(\cos(\pi x)\right)\cdot\left(-\sin(\pi x)\cdot\underbrace{\frac{d}{dx}(\pi x)}_{\pi}\right)$$

$$-2\pi\cos(\pi x)\sin(\pi x) \overset{set}{=} 0.$$

$\cos\pi x = 0$

$\Rightarrow \pi x = (2k+1)\cdot\frac{\pi}{2} \quad k\in\mathbb{Z}$

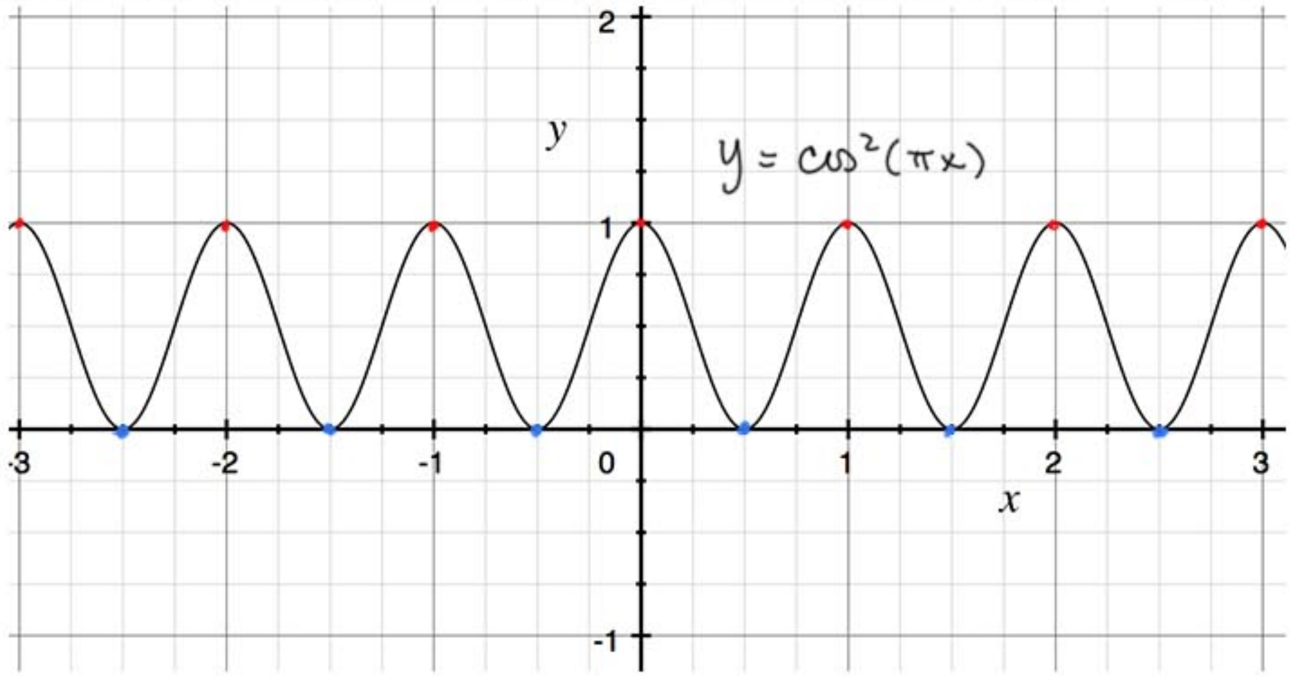
$x = \frac{2k+1}{2} \quad k\in\mathbb{Z}$ all halves

$\sin\pi x = 0$

$\pi x = k\pi \quad k\in\mathbb{Z}$

$\Rightarrow x = k \quad k\in\mathbb{Z}$

all integers



Ex. If $y = 3\sin^2(g(x))$ with $g(1) = \frac{\pi}{6}$ and $g'(1) = 2$, find $y'(1)$.

$$y = 3\left(\sin(g(x))\right)^2$$

$$y' = 3 \cdot 2\left(\sin(g(x))\right)^1 \cdot \frac{d}{dx}\left(\sin(g(x))\right)$$

$$= 6\sin(g(x)) \cdot \cos(g(x)) \cdot g'(x).$$

$$y'(1) = 6\sin(g(1)) \cdot \cos(g(1)) \cdot g'(1)$$

$$= 6\sin\left(\tfrac{\pi}{6}\right)\cos\left(\tfrac{\pi}{6}\right) \cdot 2 = 12 \cdot \tfrac{1}{2} \cdot \tfrac{\sqrt{3}}{2} = 3\sqrt{3}.$$