Math 20100
Calculus I

Lesson 11
The Product and Quotient Rules

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The Product and Quotient Rules

Recall from Lesson 10, the basic differentiation rules:

$$\frac{d}{dx}(c) = 0, \quad \frac{d}{dx}(x) = 1, \quad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}\left(cf(x)\right) = cf'(x), \quad \frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$$

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x$$

Here we learn how to take the derivative of a product of functions $f(x)g(x)$ and a quotient of functions $\frac{f(x)}{g(x)}$.

The Product Rule: $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$

Ex. $y = x^2 \sin x$

$$y' = \frac{d}{dx}(x^2)\sin x + x^2 \frac{d}{dx}(\sin x)$$

$$= 2x\sin x + x^2\cos x.$$

Ex. $y = (x^2+2x+1)(2x^3-2x)$ can multiply out first <u>or</u> use product rule

product rule: $y' = \frac{d}{dx}(x^2+2x+1)\cdot(2x^3-2x) + (x^2+2x+1)\cdot\frac{d}{dx}(2x^3-2x)$

$$= (2x+2)(2x^3-2x) + (x^2+2x+1)(6x^2-2).$$

or multiply out : $y = 2x^5 - 2x^3 + 4x^4 - 4x^2 + 2x^3 - 2x$

$$= 2x^5 + 4x^4 - 4x^2 - 2x$$

$$y' = 10x^4 + 16x^3 - 8x - 2$$

Note: $(2x+2)(2x^3-2x) + (x^2+2x+1)(6x^2-2) =$

$$= 4x^4 - 4x^2 + 4x^3 - 4x + 6x^4 - 2x^2 + 12x^3 - 4x + 6x^2 - 2$$

$$= 10x^4 + 16x^3 - 8x - 2.$$ same answer.

Ex. $y = 3\sqrt{x}\cos x$

$$y' = \frac{d}{dx}(3\sqrt{x})\cos x + 3\sqrt{x}\,\frac{d}{dx}(\cos x)$$

$$= 3\left(\tfrac{1}{2}x^{-1/2}\right)\cos x + 3\sqrt{x}\,(-\sin x)$$ ← use parentheses

$$= \frac{3\cos x}{2\sqrt{x}} - 3\sqrt{x}\sin x.$$

Ex. Suppose $h(x) = f(x)\sin x$ for some $f(x)$ with $f(\pi/3) = 4$ and $f'(\pi/3) = -2$.

Find $h'(\frac{\pi}{3})$.

We know $h'(x) = f'(x)\sin x + f(x)\frac{d}{dx}(\sin x)$

$$= f'(x)\sin x + f(x)\cos x$$

So $h'(\frac{\pi}{3}) = f'(\frac{\pi}{3})\sin(\frac{\pi}{3}) + f(\frac{\pi}{3})\cos(\frac{\pi}{3})$

$$= -2 \cdot \frac{\sqrt{3}}{2} + 4\left(\frac{1}{2}\right) = -\sqrt{3}+2.$$

The Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Ex. $y = \dfrac{3\sin x}{x^2+2x}$ $\quad y' = \dfrac{(x^2+2x)\frac{d}{dx}(3\sin x) - 3\sin x\frac{d}{dx}(x^2+2x)}{(x^2+2x)^2}$

$$= \frac{(x^2+2x)\,3\cos x - 3\sin x\,(2x+2)}{(x^2+2x)^2}$$

$$= \frac{(3x^2+6x)\cos x - (6x+6)\sin x}{(x^2+2x)^2}.$$

Ex. $y = \tan x = \dfrac{\sin x}{\cos x}$

$$y' = \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\therefore \boxed{\frac{d}{dx}(\tan x) = \sec^2 x}.$$

Ex. Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$ by using the Quotient Rule.

Work on this problem on your own

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{(\cos x)\overset{0}{\cancel{\frac{d}{dx}(1)}} - 1 \cdot \frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{-(-\sin x)}{(\cos x)^2} = \frac{\sin x}{(\cos x)^2} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \tan x .$$

$$\therefore \boxed{\frac{d}{dx}(\sec x) = \sec x \tan x} .$$

Similarly it can be shown that

$$\boxed{\frac{d}{dx}(\csc x) = -\csc x \cot x} \qquad \boxed{\frac{d}{dx}(\cot x) = -\csc^2 x} .$$

Ex. Find an equation of the tangent line to $f(x) = \tan x$ at $x = \pi/4$.

STOP Work on this problem on your own

at $x = \pi/4$, $f(\pi/4) = \tan(\pi/4) = 1$ pt $(\pi/4, 1)$

slope $= f'(\pi/4)$.

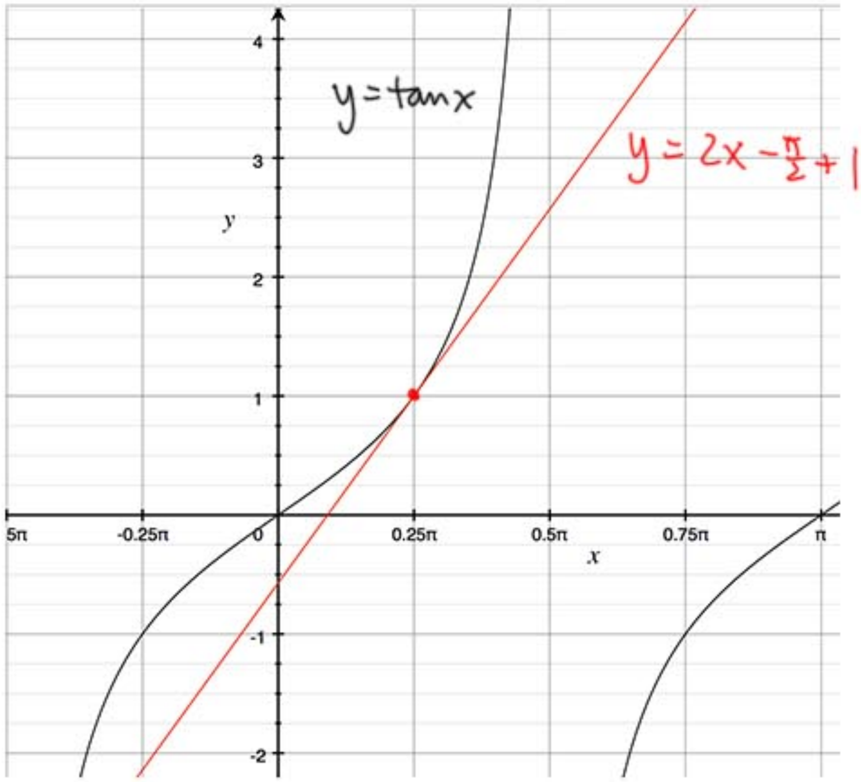
$f(x) = \tan x \quad f'(x) = \sec^2 x$

$$f'(\pi/4) = \sec^2(\pi/4) = (\sec \pi/4)^2 = \left(\frac{1}{\cos \pi/4}\right)^2$$

$$= \left(\frac{1}{\frac{\sqrt{2}}{2}}\right)^2 = \left(\frac{1}{\frac{2}{4}}\right) = \frac{4}{2} = 2 = \text{slope}$$

$$y - 1 = 2(x - \pi/4) = 2x - \frac{\pi}{2}$$

$$y = 2x - \frac{\pi}{2} + 1$$



Ex. $y = \dfrac{2x^3 \sin x}{\cos x + 1}$ product within a quotient. start with quotient rule.

$$y' = \frac{(\cos x + 1)\overbrace{\frac{d}{dx}(2x^3 \sin x)}^{\text{product rule}} - (2x^3 \sin x)\frac{d}{dx}(\cos x + 1)}{(\cos x + 1)^2}$$

$$= \frac{(\cos x + 1)\left[\frac{d}{dx}(2x^3)\cdot \sin x + 2x^3 \cdot \frac{d}{dx}(\sin x)\right] - (2x^3 \sin x)\frac{d}{dx}(\cos x + 1)}{(\cos x + 1)^2}$$

$$= \frac{(\cos x + 1)\left[6x^2 \sin x + 2x^3 \cos x\right] - (2x^3 \sin x)(-\sin x)}{(\cos x + 1)^2}$$

$$= \frac{6x^2 \sin x \cos x + 6x^2 \sin x + 2x^3 \cos^2 x + 2x^3 \cos x + 2x^3 \sin^2 x}{(\cos x + 1)^2}$$

$$= \frac{6x^2 \sin x \cos x + 6x^2 \sin x + 2x^3 \cos x + 2x^3 \overbrace{(\sin^2 x + \cos^2 x)}^{1}}{(\cos x + 1)^2}$$

$$= \frac{2x^2 (3 \sin x \cos x + 3 \sin x + x \cos x + x)}{(\cos x + 1)^2}$$

$$= \frac{2x^2 (3 \sin x + x) \cancel{(\cos x + 1)}}{(\cos x + 1)^{\cancel{2}}} = \frac{2x^2 (3 \sin x + x)}{\cos x + 1}.$$

Ex. For $f(x) = x^2(\sin x + \cos x)$, find $f'(x)$ and $f''(x)$

Work on this problem
on your own

$$f'(x) = \frac{d}{dx}(x^2) \cdot (\sin x + \cos x) + x^2 \cdot \frac{d}{dx}(\sin x + \cos x)$$

$$= 2x(\sin x + \cos x) + x^2 (\cos x - \sin x)$$

$$f''(x) = \frac{d}{dx}(2x) \cdot (\sin x + \cos x) + 2x \frac{d}{dx}(\sin x + \cos x) +$$

$$+ \frac{d}{dx}(x^2)\cdot(\cos x - \sin x) + x^2 \cdot \frac{d}{dx}(\cos x - \sin x)$$

$$= 2(\sin x + \cos x) + 2x(\cos x - \sin x) +$$

$$+ 2x(\cos x - \sin x) + x^2(-\sin x - \cos x)$$

$$= 2\sin x + 2\cos x + 2x\cos x - 2x\sin x$$

$$+ 2x\cos x - 2x\sin x - x^2\sin x - x^2\cos x$$

$$= (2 - 4x - x^2)\sin x + (2 + 4x - x^2)\cos x.$$

Ex. At which x-values is the tangent line to $f(x) = \frac{2x+2}{x-1}$ parallel to $x+y=5$?

Find equations for those tangent lines.

parallel lines have the same slope.

Parallel to $x + y = 5$

$\quad\quad\quad\quad\ \ -x \quad\ -x$

$y = -x + 5$ means slope $= -1$

so we need the x-values such that $f'(x) = -1$.

STOP Work on this problem on your own

$$f(x) = \frac{2x+2}{x-1} \qquad f'(x) = \frac{(x-1)\frac{d}{dx}(2x+2) - (2x+2)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(2) - (2x+2)(1)}{(x-1)^2}$$

$$= \frac{\cancel{2x} - 2 - \cancel{2x} - 2}{(x-1)^2} = \frac{-4}{(x-1)^2} = f'(x)$$

We need x such that $f'(x) = -1$

$$\frac{-4}{(x-1)^2} = -1$$

$$-4 = -(x-1)^2$$

$$4 = (x-1)^2$$

$$x - 1 = \pm\sqrt{4} = \pm 2. \qquad x = 3,\ x = -1.$$

$$+1 \qquad\qquad +1$$

tangent line at $x=-1$: pt $(-1, f(-1))$

$$f(-1) = \frac{2(-1)+2}{(-1)-1} = \frac{0}{-2} = 0 \qquad (-1,0)$$

$m=-1$, $\quad y-0=-1(x-(-1))$

$$y=-x-1.$$

tangent line at $x=3$: pt $(3, f(3))$

$$f(3) = \frac{2(3)+2}{3-1} = \frac{8}{2} = 4 \qquad (3,4)$$

$m=-1$, $\quad y-4=-1(x-3) = -x+3$
$\quad +4 \qquad\qquad +4$

$$y=-x+7.$$