Math 20100
Calculus I

Lesson 10
Basic Differentiation Formulas

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Basic Differentiation Formulas

In lesson 9 we saw:

For $y = f(x)$, we can denote the derivative function by:

$$f'(x) \qquad y' \qquad \frac{dy}{dx} \qquad \frac{df}{dx}$$

and the second derivative:

$$f''(x) \qquad y'' \qquad \frac{d^2y}{dx^2} \qquad \frac{d^2f}{dx^2}$$

When evaluating at $x=a$, we write

$$f'(a) \qquad y'(a) \qquad \left.\frac{dy}{dx}\right|_{x=a} \qquad \left.\frac{df}{dx}\right|_{x=a}.$$

and $\frac{d}{dx}(\ )$ = "the derivative of ( ) with respect to x"

So consider the constant function $f(x) = c$. What is $f'(x)$?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} =$$

$$= \lim_{h \to 0} \frac{0 \leftarrow \text{zero}}{h \leftarrow \text{not zero, but approaching zero}} =$$

$$= \lim_{h \to 0} 0 = 0.$$

$\therefore$ $\frac{d}{dx}(c) = 0$ the derivative of a constant function is equal to zero.

makes sense, constant functions don't change. so their rate of change = 0.

Consider the function $f(x) = x$.

we know its slope = 1 at any x-value,

so $f'(x) = 1$. ie, $\frac{d}{dx}(x) = 1$.

this can also be proven using the limit definition (see text). Actually all of the basic differentiation formulas come from using the limit definition.

$$\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$$ "the Power Rule"

Ex. $y = x^5$ $y' = 5x^4$

$g(x) = x^2$ $g'(x) = 2x$

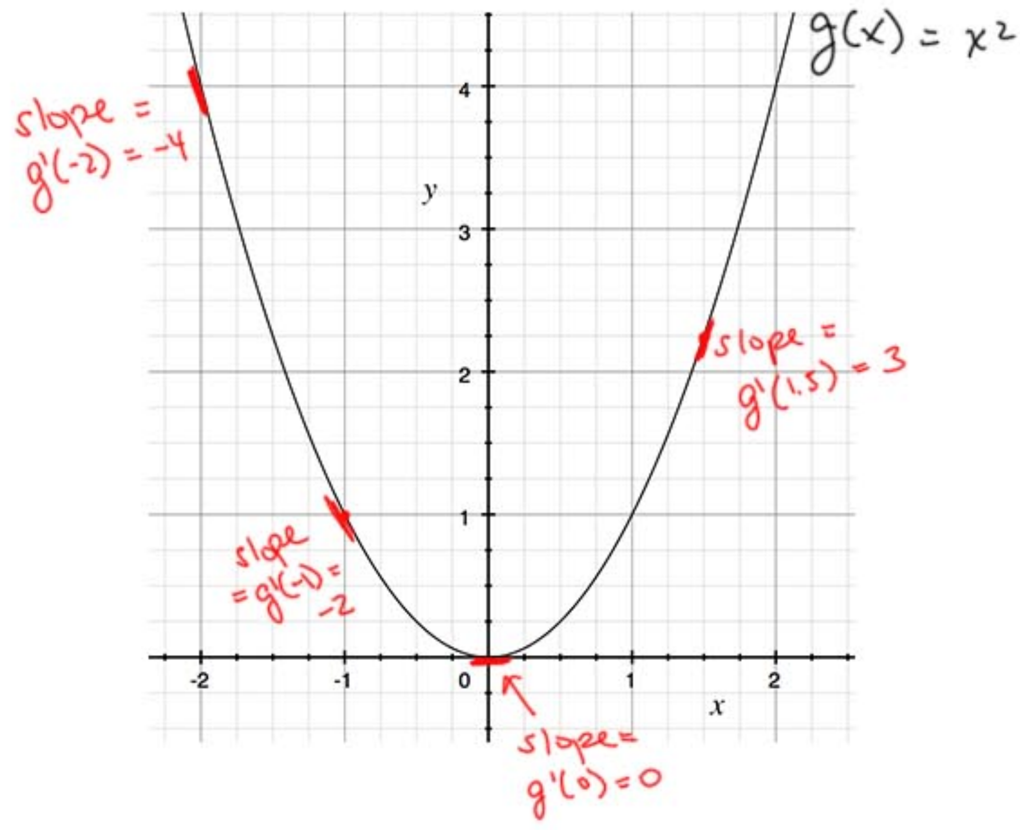
$h(x) = x^{1/2}$ $h'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$

remember the derivative function gives the slope of the original function at any x-value.

So for $g(x) = x^2$ above and $g'(x) = 2x$

slope at any point on the parabola $g(x) = x^2$

is $2 \cdot$ the x-value:



$$\frac{d}{dx}\left(c\,f(x)\right) = c\,\frac{d}{dx}\left(f(x)\right) = c\,f'(x)$$

Ex. $\frac{d}{dx}\left(5x^2\right) = 5 \cdot \frac{d}{dx}\left(x^2\right) = 5 \cdot (2x) = 10x$.

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}\left(f(x)\right) \pm \frac{d}{dx}\left(g(x)\right) = f'(x) \pm g'(x)$$

Ex. $y = 3x^5 + x$

$$y' = 3 \cdot 5x^4 + 1 = 15x^4 + 1.$$

Ex. $h(x) = \sqrt{x} - 4 = x^{1/2} - 4$

$$h'(x) = \frac{1}{2}x^{-1/2} - 0 = \frac{1}{2\sqrt{x}}.$$

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

Ex. $y = \sin x$

$$y' = \frac{d}{dx}(\sin x) = \cos x$$

$$y'' = \frac{d}{dx}(\cos x) = -\sin x$$

$$y''' = \frac{d}{dx}(-\sin x) = -\cos x$$

$$y^{(4)} = \frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x.$$

More examples:

Ex. $f(x) = 3x^3 - 4x + 2$

$f'(x) = 3 \cdot 3x^2 - 4 \cdot 1 + 0 = 9x^2 - 4$

Ex. $g(x) = x^{10} + 3^{10}$

3^{10} is a constant, don't be fooled!

$g'(x) = 10x^9 + 0$

$= 10x^9$.

Ex. $y = \frac{1}{x^2} - \frac{2}{\sqrt{x}}$

must write to fit our derivative rules

$= x^{-2} - 2x^{-1/2}$

then $y' = -2x^{-3} - 2(-\tfrac{1}{2})x^{-3/2}$

$= \frac{-2}{x^3} + \frac{1}{x^{3/2}}$

write without negative exponents.

Ex. $y = \sin t + \pi \cos t$

$y' = \cos t + \pi(-\sin t)$ ← use parentheses! no magical subtraction!

$= \cos t - \pi \sin t$

note: can write $\frac{dy}{dt} = \cos t - \pi \sin t$

↖ variable t

Ex. $h(x) = \dfrac{x^4 - 2x^2 + 1}{\sqrt{x}}$

we need to make this fit our derivative rules. We don't have a rule for quotients (yet).

$= \dfrac{x^4}{\sqrt{x}} - \dfrac{2x^2}{\sqrt{x}} + \dfrac{1}{\sqrt{x}}$

$= x^{7/2} - 2x^{3/2} + x^{-1/2}$ ← still $h(x)$

then $h'(x) = \frac{7}{2} x^{5/2} - 2 \cdot \frac{3}{2} x^{1/2} + -\frac{1}{2} x^{-3/2}$

$= \frac{7}{2} x^{5/2} - 3x^{1/2} - \dfrac{1}{2x^{3/2}}$.

Ex. $f(x) = (x+5)(2x^2+1)$

Work on this problem
on your own

rewrite $f(x)$ to fit our derivative rules

$$f(x) = 2x^3 + 10x^2 + x + 5$$

then $f'(x) = 2 \cdot 3x^2 + 10 \cdot 2x^1 + 1 + 0$

$$= 6x^2 + 20x + 1.$$

Ex. Where does $g(x) = x^2 - 4x + 2$ have a $\underbrace{\text{horizontal tangent line}}_{\text{slope} = 0}$?

need x such that $\underbrace{g'(x)}_{\text{slope of } g} = 0$

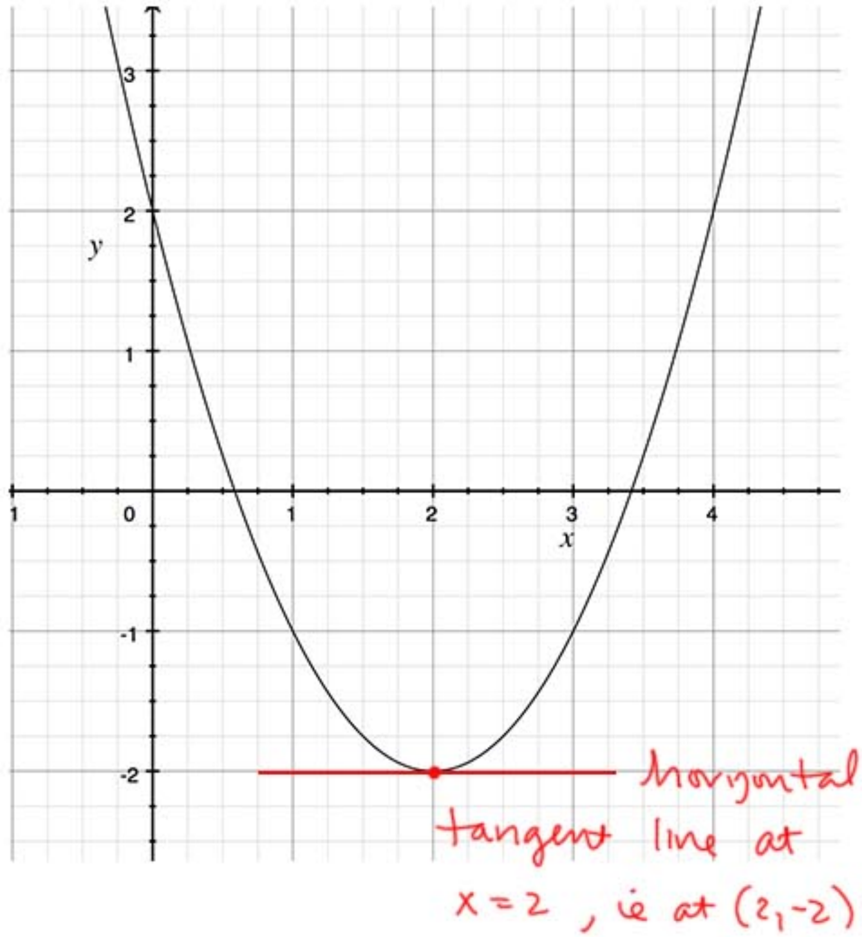
$$g'(x) = 2x - 4 \overset{\text{set}}{=} 0$$

$$2x - 4 = 0$$
$$2x = 4$$
$$x = 2$$

"where ..."

always answer with an x-value or the (x, y) pair

$$(2, g(2))$$
$$= (2, -2)$$



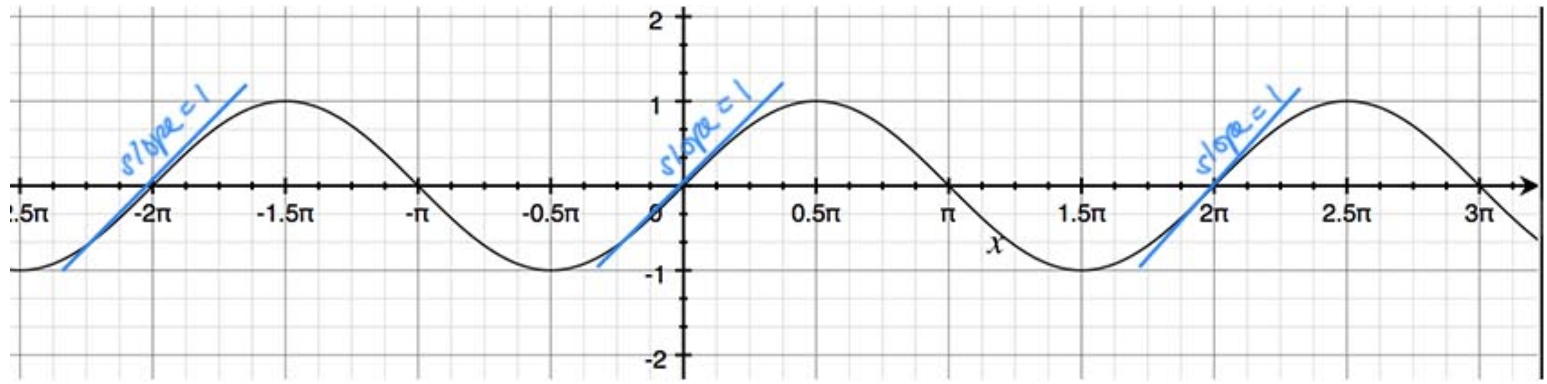
Ex. Where does $y = \sin x$ have a tangent line with slope = 1 ?

need x such that $y'(x) = 1$

$$y'(x) = \cos x \overset{set}{=} 1$$

$$x = 0, 2\pi, 4\pi, -2\pi, -4\pi, \ldots$$

$x = 2\pi k$ for $k \in \mathbb{Z}$ (integers).



Ex. Where does $f(x) = 2x^{1/3} + 1$ have a $\underbrace{\text{vertical tangent line}}_{\text{infinite slope}}$?

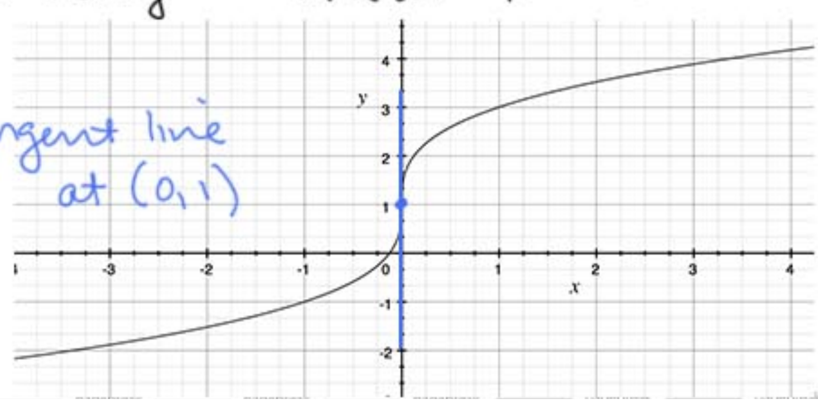
need x such that $f'(x) = \frac{\text{nonzero}}{\text{zero}}$

$$f'(x) = 2 \cdot \tfrac{1}{3} x^{-2/3} + 0 = \frac{2}{3x^{2/3}}$$

at $x = 0$, $f'(x) = \frac{2}{0}$ infinite behavior

and $\therefore$ f has a vertical tangent line at $x = 0$.

vertical tangent line at (0,1)



Ex. Find an equation of the tangent line to

$f(x) = x^{5/3} - x^{1/3}$ at $x = 27$.

Work on this problem
on your own

at $x = 27$, $f(27) = 27^{5/3} - 27^{1/3}$

$$= \left(27^{1/3}\right)^5 - 27^{1/3}$$

$$= 3^5 - 3 = 243 - 3 = 240.$$

point $(27, 240)$

slope $m = f'(27)$.

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{1}{3}x^{-2/3}$$

$$= \frac{5}{3}x^{2/3} - \frac{1}{3x^{2/3}}$$

$$f'(27) = \frac{5}{3}(27)^{2/3} - \frac{1}{3(27)^{2/3}}$$

$(27)^{2/3} = \left((27)^{1/3}\right)^2$
$= 3^2 = 9$

$= \frac{5}{3}(9) - \frac{1}{3(9)} = 15 - \frac{1}{27} = \frac{404}{27}$

$$\begin{array}{r} \overset{3}{27} \\ \underline{15} \\ 135 \\ \underline{270} \\ 405 \end{array}$$

$y - 240 = \frac{404}{27}(x - 27)$

$y - 240 = \frac{404}{27}x - 404$

$\quad +240 \qquad\qquad +240$

$y = \frac{404}{27}x - 164.$

Ex. Find an equation of the $\underbrace{\text{normal line}}_{}$ to

Perpendicular to the function and tangent line

$g(x) = \cos x - 2x + 1$ at $x = \pi$.

at $x = \pi$, $g(\pi) = \cos\pi - 2\pi + 1$

$= -1 - 2\pi + 1 = -2\pi.$

point $(\pi, -2\pi)$

Slope $m = -\frac{1}{g'(\pi)}$ since this is a normal line

$g'(x) = -\sin x - 2$

$$g'(\pi) = -\sin\pi - 2 = 0 - 2 = -2.$$

$$\therefore \ m = \frac{-1}{-2} = \frac{1}{2}.$$

$$y - (-2\pi) = \frac{1}{2}(x - \pi)$$

$$y + 2\pi = \frac{1}{2}x - \frac{\pi}{2}$$
$$-2\pi \qquad\qquad -2\pi$$

$$y = \frac{1}{2}x - \frac{5\pi}{2}.$$

Recall from Lesson 8, for a distance function $s(t)$, The rate of change (derivative) gives velocity. And The rate of change of velocity gives acceleration.

$s(t)$ distance function

$s'(t) = v(t)$ velocity function

$s''(t) = v'(t) = a(t)$ acceleration function

Ex. If a football is thrown from a height of 6 ft with an intial upward velocity of 50 ft/sec, the height of the football (neglecting air resistance) is given by $s(t) = 6 + 50t - 16t^2$.

a) what is the maximum height of the football?

b) what is the velocity of the ball when it is caught at a height of 5 ft?

a) to find the maximum height, we realize that this occurs when the upward velocity is equal to zero. So we find $s'(t)$ and set = 0 then we plug back into $s(t)$ to find the height at that time:

$$s(t) = 6 + 50t - 16t^2$$

$$s'(t) = 50 - 32t \overset{set}{=} 0$$

$$50 - 32t = 0 \quad \Rightarrow \quad t = \frac{50}{32} = \frac{25}{16} \text{ sec.}$$

$$\approx 1.56 \text{ sec.}$$

$$s\left(\frac{25}{16}\right) = 6 + 50\left(\frac{25}{16}\right) - 16\left(\frac{25}{16}\right)^2$$

$$16\left(\frac{25}{16}\right)^2 = \frac{\cancel{16}\cdot 25^2}{\cancel{16}\cdot 16}$$

$$= 6 + \underbrace{\frac{1250}{16} - \frac{625}{16}}$$

$$= \frac{96}{16} + \frac{625}{16} = \frac{721}{16} \approx 45.06 \text{ ft.}$$

b) first we have to find the time at which the height = 5 ft. then find the veolicty at that time.

$$\text{set } s(t) = 5: \quad 6 + 50t - 16t^2 = 5$$

$$1 + 50t - 16t^2 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-50 \pm \sqrt{2500 - 4(1)(-16)}}{-32}$$

$$= \frac{-50 \pm \sqrt{2564}}{-32} \Rightarrow \begin{array}{l} t \approx 3.145 \\ t \approx -0.020 \end{array}$$

$\leftarrow$ reject, before ball is thrown.

$$v(t) = s'(t) = 50 - 32t$$

$$v(3.145) = 50 - 32(3.145) = -50.64$$

$\therefore$ The velocity of the ball when it is caught is

approximately -50.64 ft/s.

indicates height is decreasing.

can say speed ≈ 50.64 ft/s.

|velocity|