Math 20100
Calculus I

Lesson 7
Limits Involving Infinity

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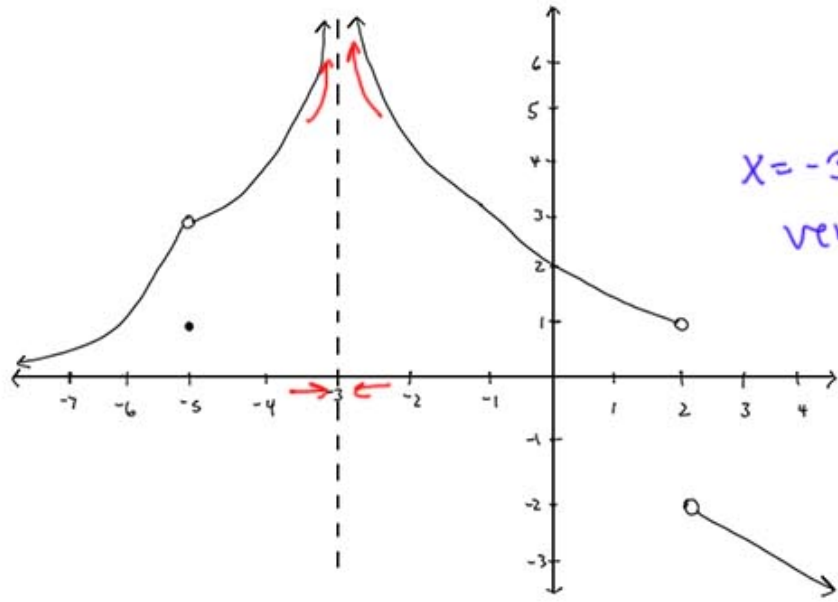
Limits Involving Infinity

We introduced This topic in Lesson 3, and now we give more details.

Infinite Limits

Recall:

from Lesson 3



$x=-3$ is a vertical asymptote.

We said that $\lim\limits_{x \to -3} f(x)$ does not exist as a real number,

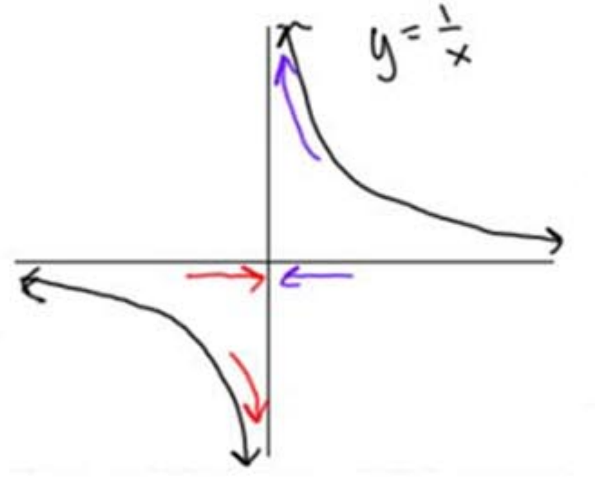
but we write $\lim\limits_{x \to -3} f(x) = \infty$ to denote That the function values increase without bound.

We also saw

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

and $\lim_{x \to 0} \frac{1}{x}$ DNE.

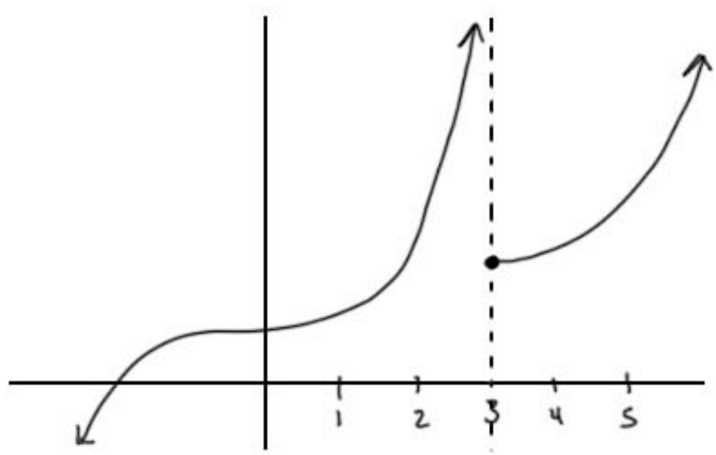


x=0 is a vertical asymptote

In Lesson 5 we saw an infinite discontinuity,

defined at x=a when:

$\lim_{x \to a} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$



$\lim_{x \to 3^-} f(x) = \infty$

so we have an infinite discontinuity at x=3.

x=3 is a vertical asymptote of this graph

Def. For $f(x)$ with

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \pm\infty,$$

we say the line $x = a$ is a <u>vertical asymptote</u> of the graph of $y = f(x)$.

Recall from Lesson 3, we said $\frac{\text{nonzero}}{\text{zero}}$ is indicative of infinite behavior.

So, if we are given the algebraic representation of a function and want to find vertical asymptotes, we find the x-values for which $f(x)$ would be $\frac{\text{nonzero}}{\text{zero}}$.

Ex. Find any vertical asymptotes of

$$f(x) = \frac{2x^2+6x}{x^2+2x-3}$$

we want to find all x-values such that

$f(x)$ would be $\frac{\text{nonzero}}{\text{zero}}$.

So set $x^2+2x-3=0$

$(x+3)(x-1)=0$

$x+3=0 \quad x-1=0$

$x=-3 \quad x=1$

for $x=1$, $f(x) = \frac{2(1)^2+6(1)}{0} = \frac{\text{nonzero}}{\text{zero}}$, infinite behavior

$\therefore$ $x=1$ is a vertical asymptote of $f(x)$.

for $x=-3$, $f(x) = \frac{2(-3)^2+6(-3)}{0} = \frac{0}{0}$

$\frac{0}{0}$ is an indeterminate form, means we need more info.

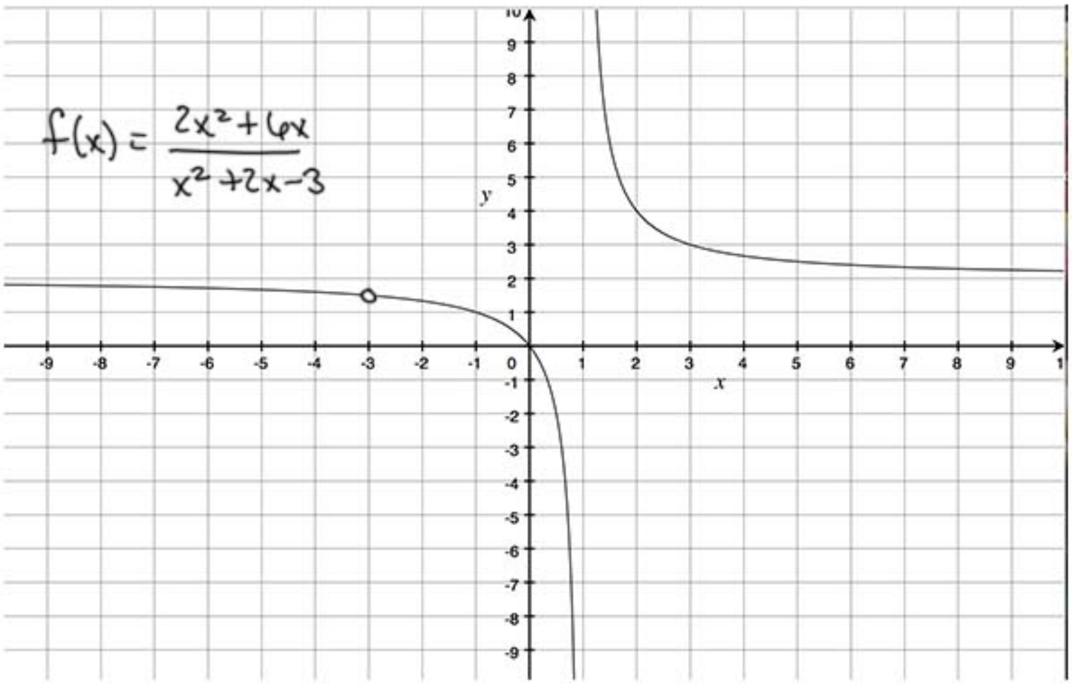
Examine: $f(x) = \frac{2x^2+6x}{x^2+2x-3} = \frac{2x\cancel{(x+3)}}{\cancel{(x+3)}(x-1)} = \frac{2x}{x-1}$ for $x \neq -3$

So the graph of $f(x)$ is the graph of $g(x) = \frac{2x}{x-1}$ <u>except</u>

at the point $x=-3$.

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} g(x) = \frac{2(-3)}{-3-1} = \frac{-6}{-4} = \frac{3}{2}$$ ← a hole in the graph at $\left(-3, \frac{3}{2}\right)$

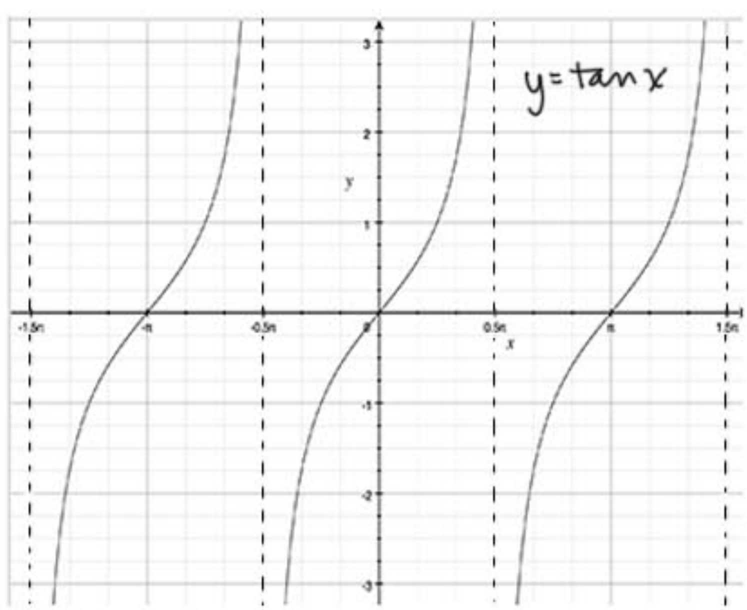
NOT an asymptote,
a removable discontinuity.



Ex. Find the vertical asymptotes of $f(x) = \tan 2x$

we know the graph of $y = \tan x$ has vertical asymptotes where $\cos x = 0$, ie. $(2k+1)\frac{\pi}{2}$ $k \in \mathbb{Z}$ ← integers

(odd multiples of $\frac{\pi}{2}$)



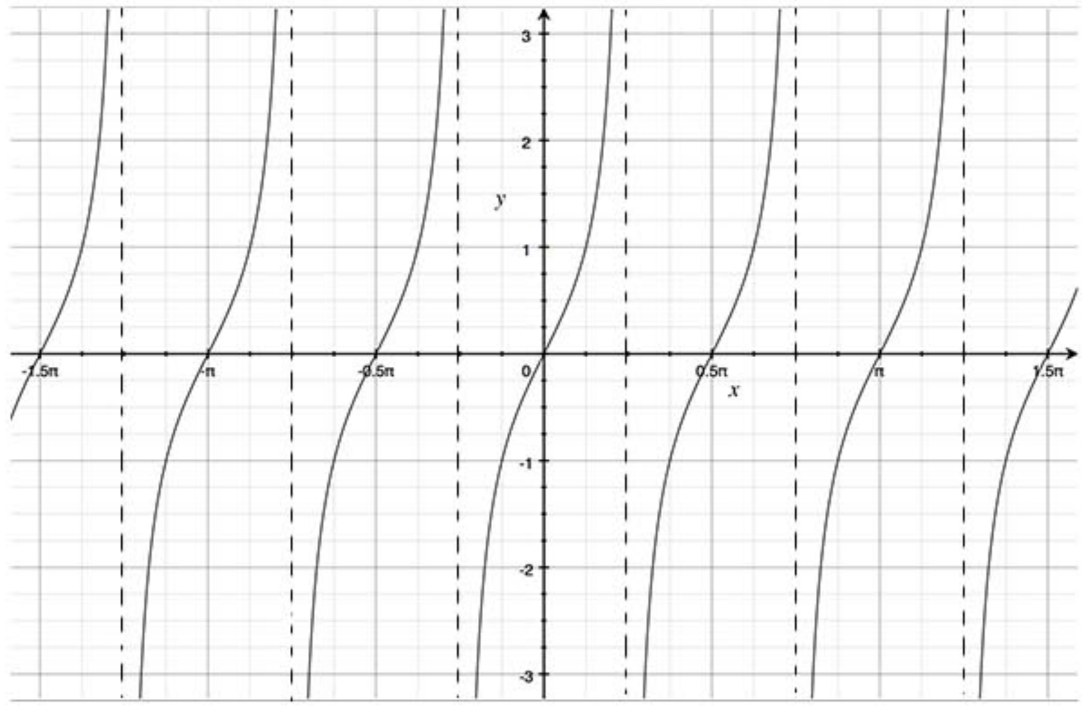
The graph of $f(x) = \tan 2x$ has a vertical compression by a factor of 2, so the vertical asymptotes will be at odd multiples of $\frac{\pi}{4}$:

i.e. where

$\cos 2x = 0$

$(2k+1)\frac{\pi}{4} \quad k \in \mathbb{Z}$

integers.



Finding Infinite Limits

$$\lim_{x \to 2^-} \frac{-3x}{x-2}$$

we know here we have $\frac{\text{nonzero}}{\text{zero}}$,

but is the limit $+\infty$ or $-\infty$?

Choose a number near $x = 2$ but from the left,

ie $x = 1.9$, plug in, and determine the sign:

$$\frac{-3(1.9)}{1.9-2} = \frac{-}{-} = + \quad \therefore \lim_{x \to 2^-} \frac{-3x}{x-2} = +\infty$$

Similarly,

$\lim_{x \to 2^+} \frac{-3x}{x-2}$ we know we have an infinite limit,

Choose 2.1 + plug in: $\frac{-3(2.1)}{2.1-2} = \frac{-}{+} = -$

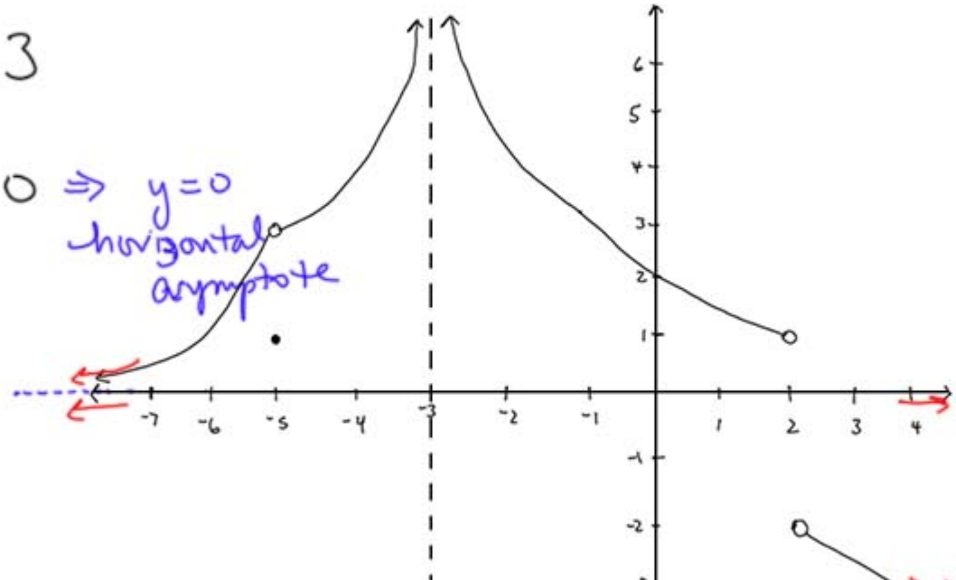
$$\therefore \lim_{x \to 2^+} \frac{-3x}{x-2} = -\infty.$$

Limits at Infinity

Recall: lesson 3

we said $\lim_{x \to -\infty} f(x) = 0 \Rightarrow y=0$ horizontal asymptote

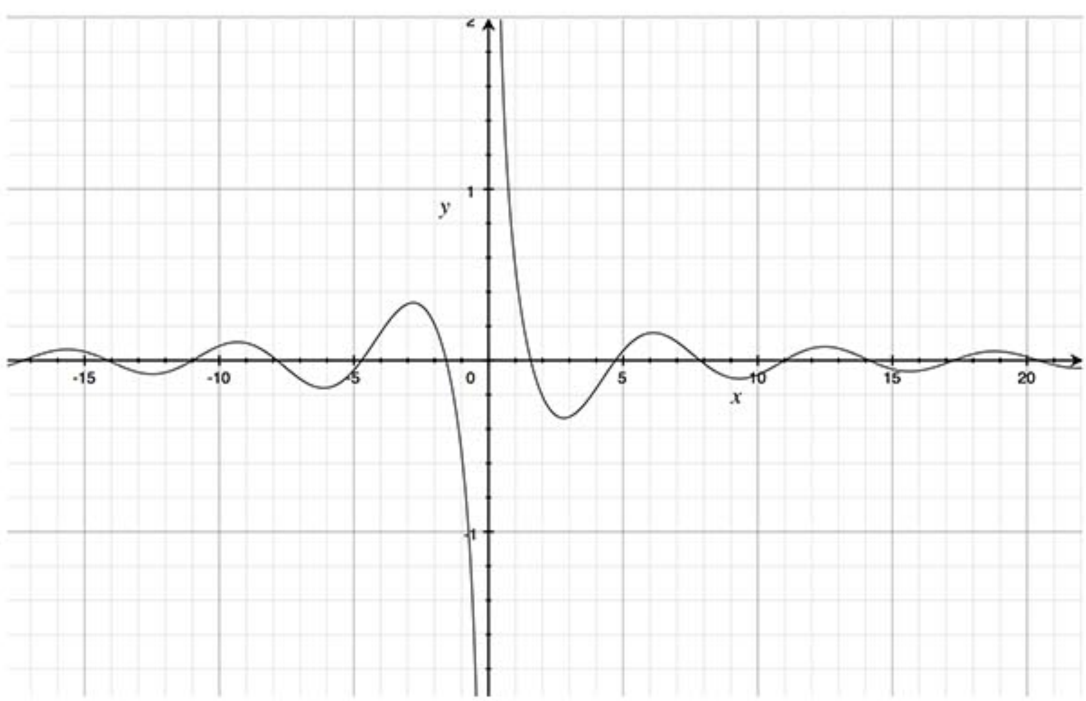
+ $\lim_{x \to \infty} f(x) = -\infty$



We call limits as $x \to \infty$ or $x \to -\infty$ "end behavior" of the graph.

Def. The line $y = L$ is <u>horizontal asymptote</u> of f if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Ex. $f(x) = \dfrac{\cos x}{x}$ we have $\lim_{x\to\pm\infty} f(x) = 0$



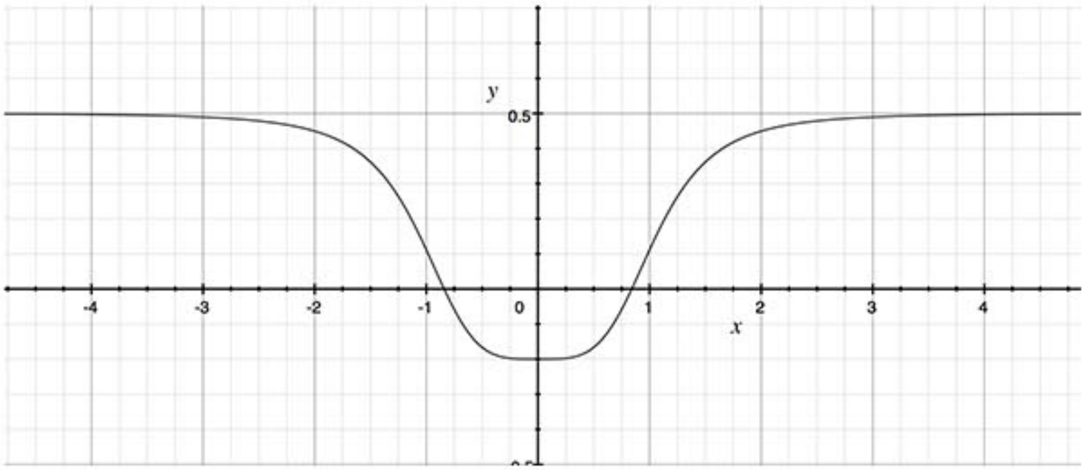
horizontal asymptote at $y = 0$.

*Note: a function <u>can</u> cross a horizontal asymptote.

Notice there is also a vertical asymptote at $x = 0$

$$\frac{\cos(0)}{0} = \frac{1}{0} \quad \frac{\text{nonzero}}{\text{zero}} \Rightarrow \text{infinite behavior.}$$

Ex. $f(x) = \dfrac{2x^4 - 1}{4x^4 + 5}$ horizontal asymptote at $y = \frac{1}{2}$



since $\lim_{x \to \pm\infty} \dfrac{2x^4 - 1}{4x^4 + 5} = \dfrac{1}{2}$.

Limits at Infinity of Rational Functions

We use $\lim_{x \to \pm\infty} \dfrac{1}{x^n} = 0$ for n a positive integer

dividing by larger and larger numbers makes the fraction approach zero.

Ex. above $\lim_{x \to \infty} \dfrac{2x^4 - 1}{4x^4 + 5} \cdot \dfrac{\frac{1}{x^4}}{\frac{1}{x^4}}$

* divide by the highest power found in the denominator

$= \lim_{x \to \infty} \dfrac{2 - \cancel{\frac{1}{x^4}} \to 0}{4 + \cancel{\frac{5}{x^4}} \to 0} = \dfrac{2}{4} = \dfrac{1}{2}$.

Ex. Find the horizontal asymptotes of $f(x)=\dfrac{3x^3-2x^2+1}{x^5-x^3+2}$

Work on this problem
on your own

$$\lim_{x\to\infty}\frac{3x^3-2x^2+1}{x^5-x^3+2}\cdot\frac{\frac{1}{x^5}}{\frac{1}{x^5}}=\lim_{x\to\infty}\frac{\frac{3}{\cancel{x^2}}\nearrow^{0}-\frac{2}{\cancel{x^3}}\nearrow^{0}+\frac{1}{\cancel{x^5}}\nearrow^{0}}{1-\frac{1}{\cancel{x^2}}\nearrow^{0}+\frac{2}{\cancel{x^5}}\nearrow^{0}}=\frac{0}{1}=0$$

$\therefore$ $y=0$ is a horizontal asymptote of the graph of $f(x)$.

for $\lim\limits_{x\to-\infty}f(x)$, same computation. $\therefore$ $y=0$ is the only horizontal asymptote.

Ex. $\lim\limits_{x\to-\infty}\dfrac{x^4+2x^3-1}{2x^3-3x+2}$

Work on this problem
on your own

$$\lim_{x\to-\infty}\frac{x^4+2x^3-1}{2x^3-3x+2}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}=\lim_{x\to-\infty}\frac{x+2-\frac{1}{\cancel{x^3}}\nearrow^{0}}{2-\frac{3}{\cancel{x^2}}\nearrow^{0}+\frac{2}{\cancel{x^3}}\nearrow^{0}}=-\infty$$

Ex. $\lim\limits_{x \to \infty} \left(\sqrt{1+4x^2} - \sqrt{x} \right)$ looks like $\infty - \infty$

indeterminate form,

we need more information

$\Rightarrow$ we need to manipulate the expression mathematically to be more convenient

$$\lim_{x \to \infty} \left(\sqrt{1+4x^2} - \sqrt{x} \right) \cdot \frac{\left(\sqrt{1+4x^2} + \sqrt{x} \right)}{\left(\sqrt{1+4x^2} + \sqrt{x} \right)}$$

rationalize the numerator by multiplying by the conjugate over itself

$$= \lim_{x \to \infty} \frac{(1+4x^2) - (x)}{\sqrt{1+4x^2} + \sqrt{x}} = \lim_{x \to \infty} \frac{4x^2 - x + 1}{\sqrt{1+4x^2} + \sqrt{x}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$\sqrt{x^2} = x$ for $x \geq 0$ highest power in denominator

$$= \lim_{x \to \infty} \frac{4x - 1 + \cancel{\frac{1}{x}}^{\,0}}{\sqrt{\cancel{\frac{1}{x^2}}^{\,0} + 4} + \sqrt{\cancel{\frac{1}{x}}}^{\,0}} = \infty .$$

$\frac{1}{x} = \sqrt{\frac{1}{x^2}}$ for $x \geq 0$