Math 20100
Calculus I

Lesson 06
Continuity

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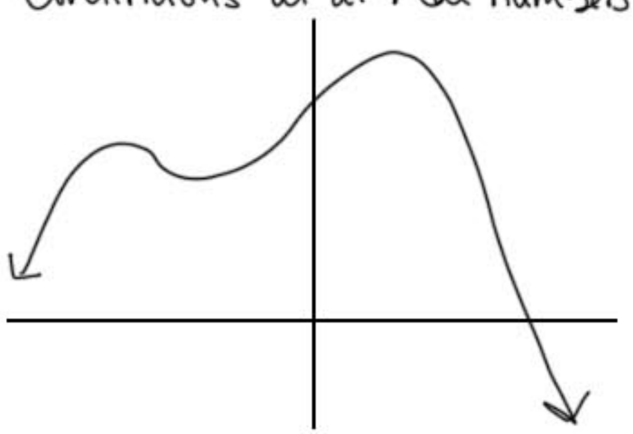
Continuity

<u>Definition</u>: A function f is said to be <u>continuous</u> at $x = a$ if $\boxed{\lim_{x \to a} f(x) = f(a)}$.

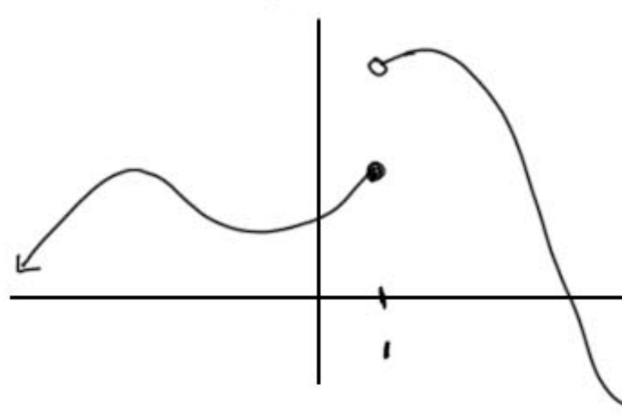
If f is not continuous at $x = a$, we say f has a <u>discontinuity</u> at $x = a$.

Intuitively continuity means we can draw the graph of $y = f(x)$ near $x = a$ without picking up our pens.

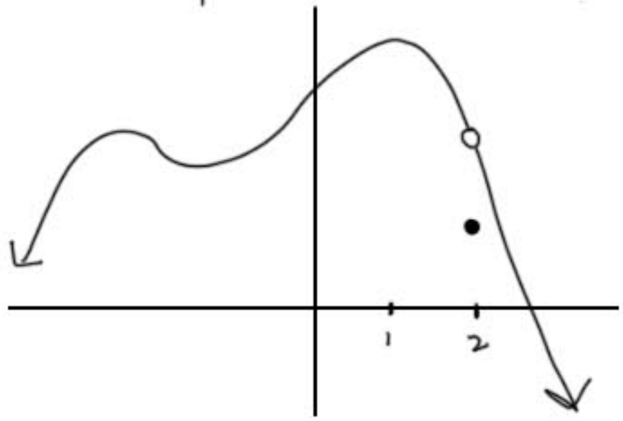
continuous at all real numbers



continuous for all $x \neq 1$.



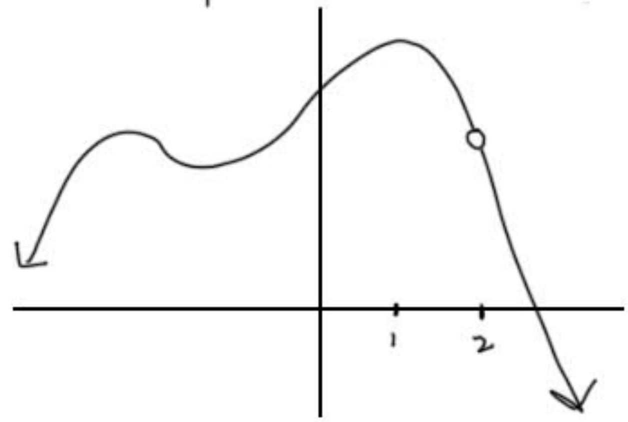
discontinuity at $x = 1$ because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, ie $\lim_{x \to 1} f(x)$ DNE continuous for all $x \neq 2$



discontinuity at $x = 2$

because $\lim_{x \to 2} f(x) \neq f(2)$

continuous for all $x \neq 2$



discontinuity at $x = 2$

because $f(2)$ DNE.

Note #1: Saying $\lim_{x \to a} f(x) = f(a)$ is equivalent

to the following three statements:

1) $\lim_{x \to a} f(x)$ exists (two-sided limit, not infinite)

2) $f(a)$ exists ($x = a$ is in the domain of f)

3) $\lim_{x \to a} f(x) = f(a)$

When explaining discontinuities, it is often helpful to remember the three separate statements.

Note #2: f being continuous at $x=a$ means we can push/pull the limit through the function:

$$\lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right) = f(a).$$

Says to look at all the $f(x)$ values as the x values approach a, and take the limit of the $f(x)$ values

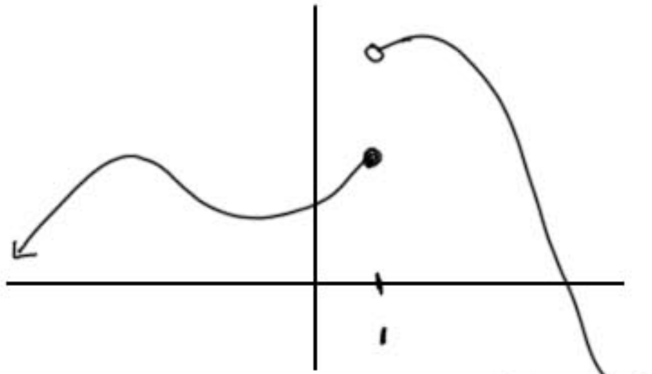
Says let x approach a, then plug that limit value, a, into the function

This comes up in Calc II a bit.

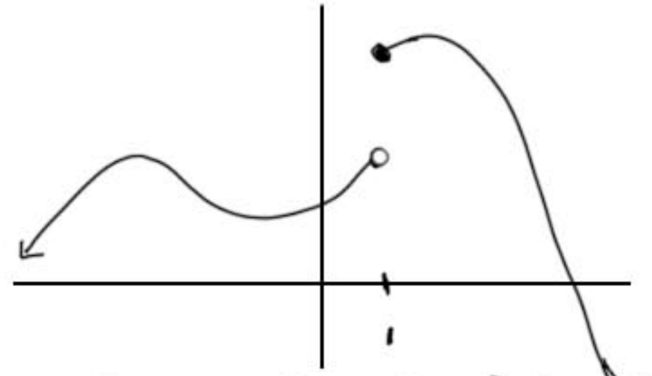
More definitions:

If $\lim_{x\to a^+} f(x) = f(a)$, f is continuous from the right.

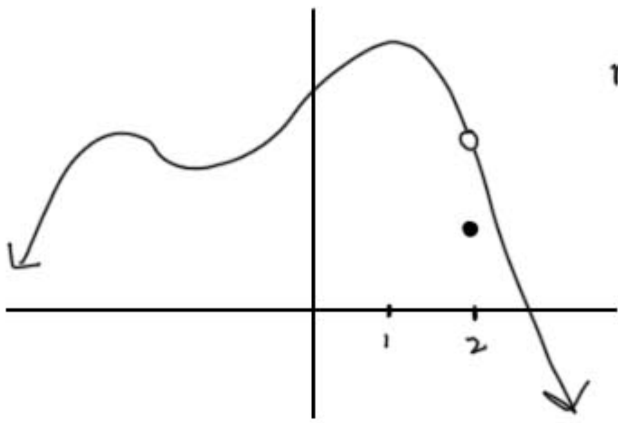
If $\lim_{x\to a^-} f(x) = f(a)$, f is continuous from the left.



continuous from the left at $x=1$, ie $\lim_{x \to 1^-} f(x) = f(1)$



continuous from the right at $x=1$, ie $\lim_{x \to 1^+} f(x) = f(1)$

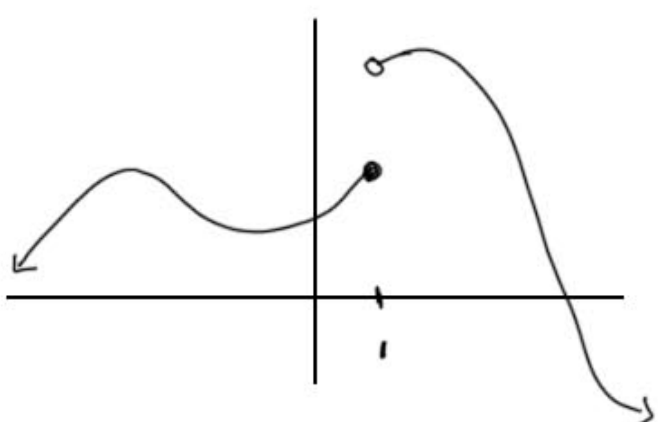


not continuous from the left or right at $x=2$

ie, $\lim_{x \to 2^-} f(x) \neq f(2)$

$\lim_{x \to 2^+} f(x) \neq f(2)$

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, f has a <u>jump discontinuity</u> at $x=a$.

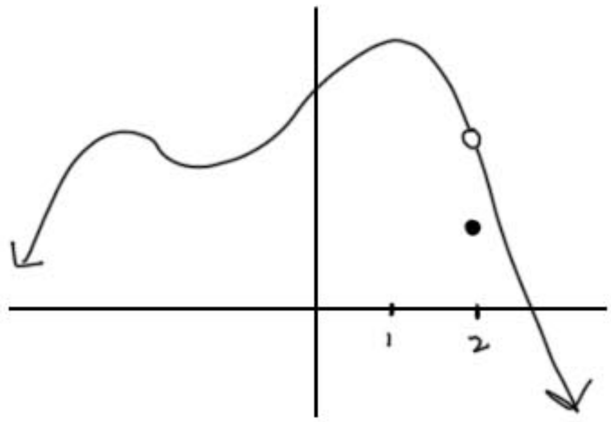


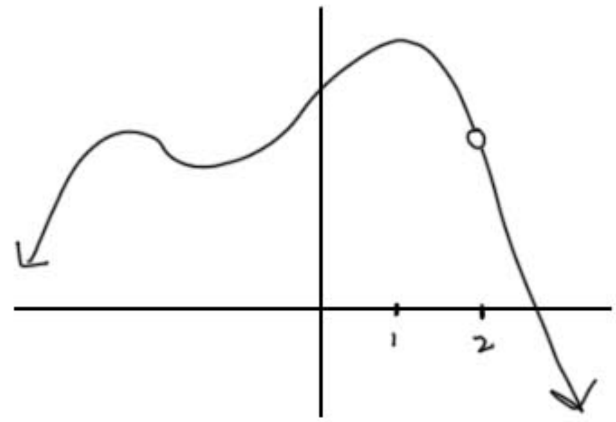
jump discontinuity at $x=1$

ie. $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$ (or perhaps $f(a)$ DNE)

f has a <u>removable discontinuity</u> at $x = a$.

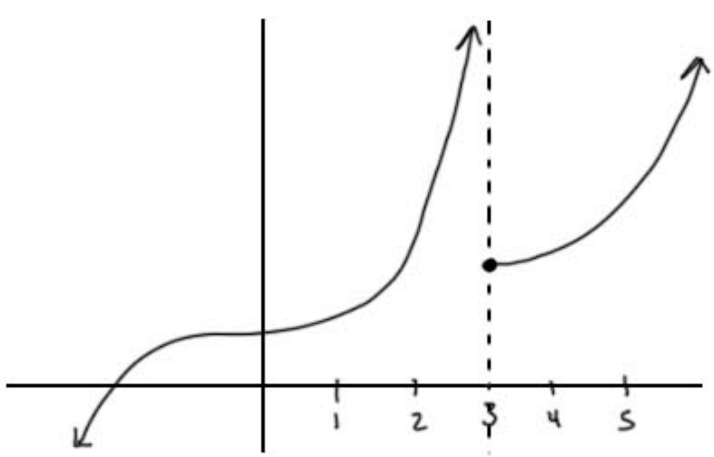




these functions have removable discontinuities at $x = 2$.

If $\lim_{x \to a} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$,

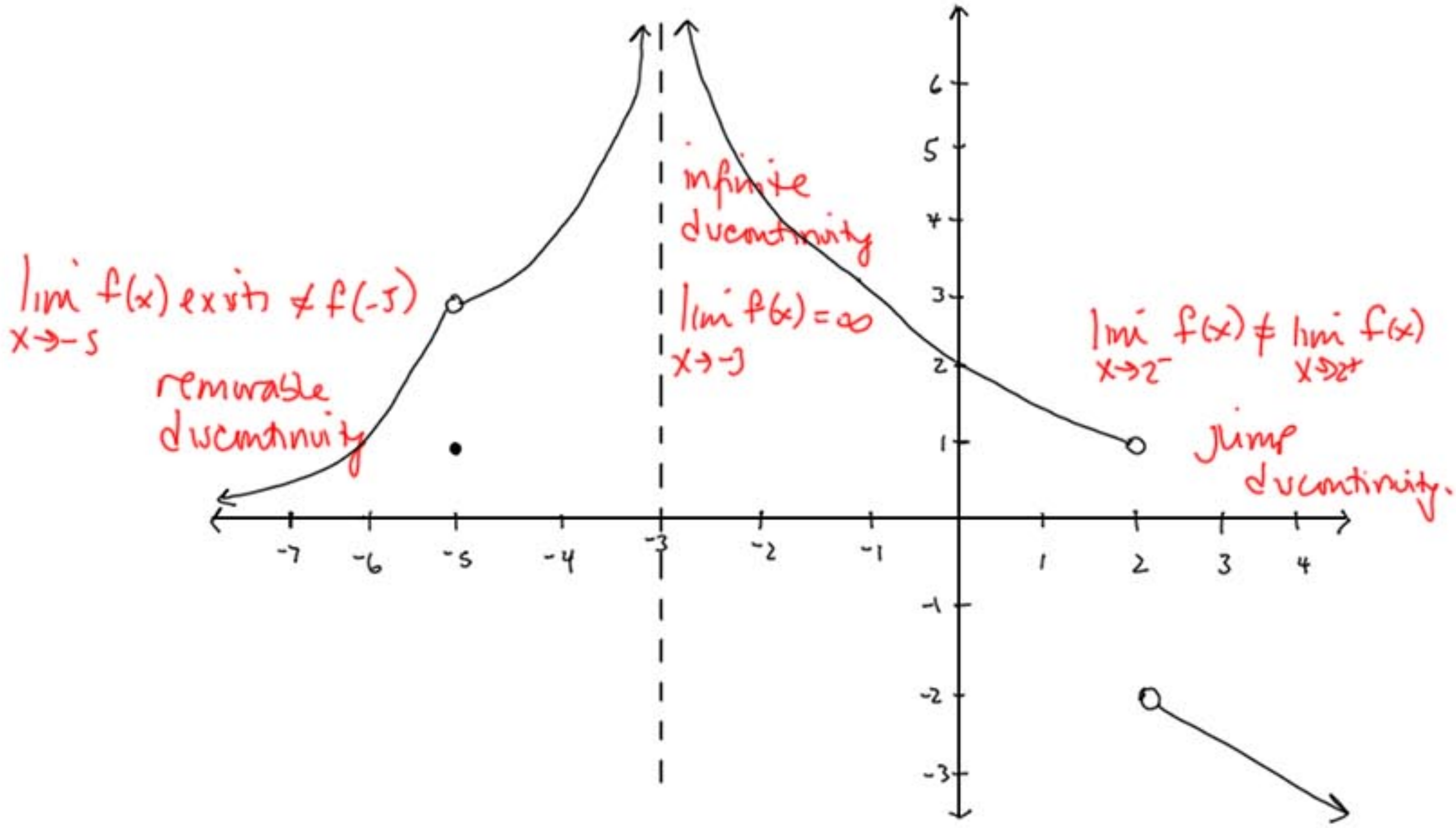
f has an <u>infinite discontinuity</u> at $x = a$.



$\lim_{x \to 3^-} f(x) = \infty$

so we have an infinite discontinuity at $x = 3$.

Ex. Find and classify the discontinuities.



STOP Work on this problem on your own

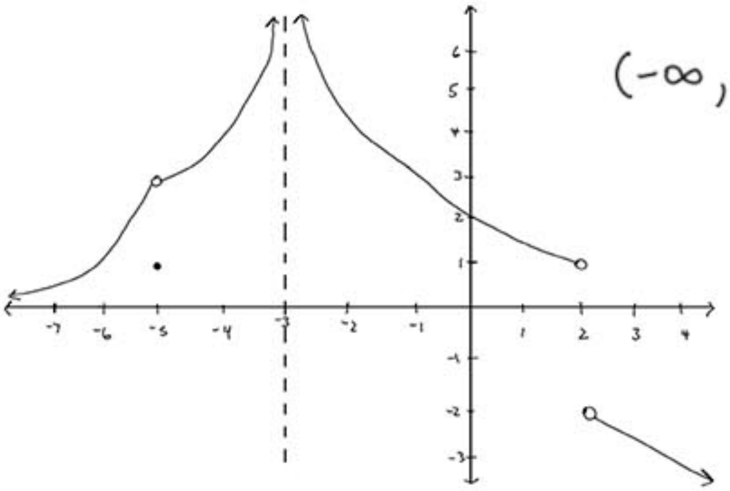
removable discontinuity at $x = -5$

infinite discontinuity at $x = -3$

jump discontinuity at $x = 2$.

Def. If f is continuous at all x values in the interval (a,b), f is said to be <u>continuous on the interval</u> (a,b).

Ex. State the intervals on which the function is continuous.



$$(-\infty,-5)\cup(-5,-3)\cup(-3,2)\cup(2,\infty)$$

Ex. State the intervals on which f is continuous:

$$f(x)=\begin{cases}7-2x & x\le -1\\ x-2 & -1<x\le 2\\ -x^2+4 & x>2\end{cases}$$

STOP Work on this problem on your own

We know that lines and parabolas are continuous for all real numbers, so we only need to check the x-values at which the function definition changes, ie. $x = -1$ and $x = 2$.

$$f(x) = \begin{cases} 7-2x & x \le -1 \\ x-2 & -1 < x \le 2 \\ -x^2+4 & x > 2 \end{cases}$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 7-2x = 7-2(-1) = 9$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x-2 = -1-2 = -3$$

$\therefore f$ is not continuous at $x = -1$. $\left(\lim_{x \to -1} f(x) \text{ DNE}\right)$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x-2 = 2-2 = 0$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} -x^2+4 = -(2^2)+4 = -4+4 = 0$$

$\lim_{x \to 2} f(x)$ exists and $= f(2)$

continuous at $x=2$.

$\therefore f$ is continuous on $(-\infty, -1) \cup (-1, \infty)$.

Based on The Limit Laws from Lesson 4, if f and g are continuous at $x=a$ and c is a constant:

$$y = f(x) + g(x) \qquad y = cf(x) \qquad y = \frac{f(x)}{g(x)} \quad g(a) \neq 0$$

$$y = f(x) - g(x) \qquad y = f(x)g(x)$$

are also continuous at $x = a$.

Also, The Direct Substitution Property we saw in Lesson 4 for polynomials, rational functions, root functions, and trig functions comes from:

Theorem: Polynomials, rational functions, root functions, and trig functions are continuous at every x-value in their domains.

Ex. Find the interval(s) on which the functions are continuous:

a) $f(x) = \dfrac{2x^2-1}{x+3}$

here $x \neq -3$

$(-\infty,-3) \cup (-3,\infty)$

b) $h(x) = \sqrt[4]{x}$

need $x \geq 0$ domain

$[0,\infty)$

continuous at $x=0$, but there are no x-values left of $x=0$ in domain

means continuous from the right at $x=0$

Also, since continuity means we can push/pull the limit through the function, we have:

Theorem: If g is continuous at a, and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

Ex. Find the interval(s) on which the functions are continuous:

a) $f(x) = \sqrt[3]{\dfrac{x+1}{x^2-1}}$

composition of continuous functions, so continuous on its domain:

$x^2-1 \neq 0 \Rightarrow x \neq \pm 1$

$(-\infty,-1) \cup (-1,1) \cup (1,\infty)$.

b) $g(x) = \sin(2x^3+1)$

composition of continuous functions, so continuous on its domain: $\mathbb{R}$

$(-\infty,\infty)$.

Ex. Use continuity to find the limit:

$$\lim_{x \to \frac{\pi}{2}} \sin^2\left(\frac{x}{2\sin x}\right) = \sin^2\left(\frac{\frac{\pi}{2}}{2(\sin\frac{\pi}{2})}\right) = \sin^2\left(\frac{\frac{\pi}{2}}{2(1)}\right)$$

$$= \sin^2\left(\frac{\pi}{4}\right) = \left(\sin\left(\frac{\pi}{4}\right)\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}.$$

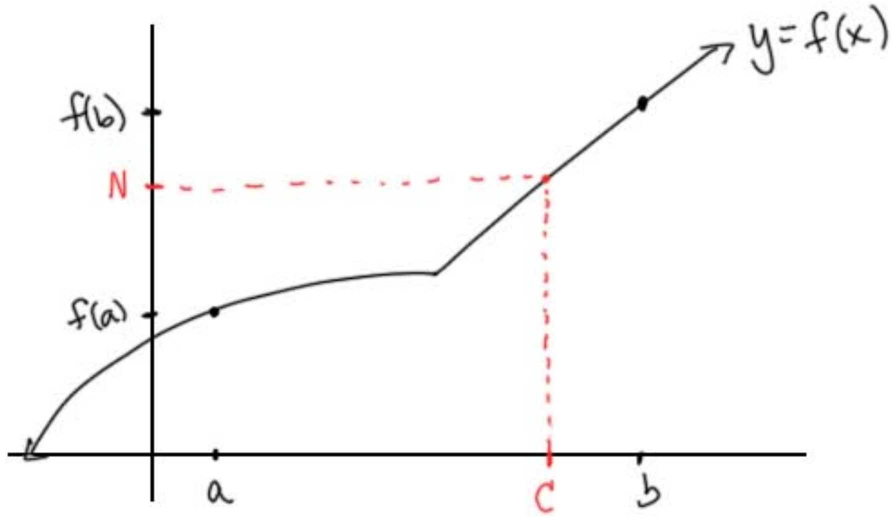
The Intermediate Value Theorem:

If f is continuous on $[a,b]$ and

N is any number between $f(a)$ & $f(b)$

$(f(a) \neq f(b))$, then there exists $c \in (a,b)$

such that $f(c) = N$.



We can use the Intermediate Value Theorem

to prove the existence of x-values that

solve a given equation:

Ex. Prove that there is a solution to

$\sin(x) = x^2 - x$ in the interval $(1,2)$.

Consider $f(x) = \sin(x) - x^2 + x$. We need to prove that there is a value $c \in (1,2)$ for which $f(c) = 0$.

look at $f(1) = \sin(1) - (1)^2 + 1 = \underbrace{\sin(1)}_{\text{positive \#}}$ one radian $\approx \frac{\pi}{3}$

also look at $f(2) = \sin(2) - 2^2 + 2 =$

2 radians, $\approx \frac{2\pi}{3}$

$$\sin(2) - 4 + 2 = \underbrace{\sin(2) - 2}_{\text{negative \#}}$$

$\therefore \quad f(2) < 0 < f(1)$, ie zero is between $f(1)$ & $f(2)$.

and by The Intermediate Value Theorem,

we know $\exists \, c \in (1,2)$ such That $f(c) = 0$.

"there exists"